UNIVERSITE PARIS-SUD

Centre D'Orsay

THESE

De Doctorat D'Etat Es Sciences Mathematiques

présentée pour obtenir le grade de

DOCTEUR ES-SCIENCES

par

Yoshinori MORIMOTO

Sujet de la Thèse :

" La propagation des singularités et l'hypoellipticité pour les équations aux dérivées partielles "

Soutenue le 1er Février 1983 *devant le Jury composé de* :

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	S. Alinhac	
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	J. Sjöstrand	
	J. Vaillant	

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à la mémoire de Monsieur Hitoshi Kumano-go,

à mes parents.

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Introduction

1ère partie

[1] Fundamental solution for a hyperbolic equation with involutive characteristics of variable multiplicity .

[2] Avec C. Iwasaki : Propagation of singularities of solutions for a hyperbolic system with nilpotent characteristics, I .

[3] On the propagation of the wave front set of a solution for a hyperbolic system .

[4] Avec C. Iwasaki : Propagation précise des singularités pour des systèmes hyperboliques à caractéristiques de multiplicité variable .

[5] Propagation du front d'onde Gevrey pour un exemple de système hyperbolique .

2ème partie

[6] On the hypoellipticity for infinitely degenerate semi-elliptic operators .

[7] On hypoelliptic operators with multiple characteristics of odd order .

INTRODUCTION

Cette thèse se compose de deux parties: La première partie est consacrée à l'étude de la propagation des singularités des solutions du problème de Cauchy hyperbolique. La deuxième traite de l'hypoellipticité des opérateurs pseudo-différentiels.

Tous les articles de la première partie se rapportent au résultat de Kumano-go et Taniguchi [K-T]. Rappelons ce résultat: Soit $\mathbb{L}$ un système hyperbolique d'ordre 1 à partie principale diagonale, c'est à dire,

$$\mathbb{L} = \begin{bmatrix} D_t + \lambda_1(t,x,D_x) & & 0 \\ & \ddots & \\ 0 & & D_t + \lambda_\ell(t,x,D_x) \end{bmatrix} + \mathbb{B}(t,x,D_x), \text{ sur } R_t \times R_x^n ,$$

où $\lambda_j(t,x,\xi) \in B_t^\infty(S^1)$ ($j = 1,\ldots,\ell$) sont à valeur réelle et $\mathbb{B}(t,x,\xi) \in B_t^\infty(S^0)$. Ici S^m désigne la classe usuelle des symboles d'opérateurs pseudo-différentiels d'ordre m . En utilisant les opérateurs integraux de Fourier à multi-phases, [K-T] ont construit la solution fondamentale du problème de Cauchy

$$\text{(P.C)} \qquad \mathbb{L}\, U = 0 \ , \ U(0,x) = G \ , \ G \in H^{-\infty} \equiv \bigcup_{s=-\infty}^{\infty} H^s \ ,$$

et ils ont donné une description de la propagation du front d'onde C^∞ de la solution

$$\text{(1)} \qquad \text{WF}\, U(t_0) \subset \Gamma(t_0 \ ; \text{WF}\, G) \ , \ t_0 > 0 \ .$$

Cette description (1) est assez compliquée bien que le système $\mathbb{L}$ soit simple et le problème (P.C) soit évidemment bien-posé. Avant de préciser la notation $\Gamma(t_0 \ ; \cdot)$, introduisons l'article [1].

L'article [1] traite du problème de Cauchy

$$\text{(P.C)'} \qquad L\, u = 0 \ , \ u(0,x) = g \ , \ g \in H^{-\infty} \ ,$$

où L est un opérateur singulier hyperbolique d'ordre élève à caractéristiques involutives de multiplicité variable. La propagation du front d'onde de ce problème a été aussi étudiée par Uhlman, Alinhac, R. Lascar et Nosmas. Dans l'article [1], on obtient la solution fondamentale pour (P.C)', en réduisant ce problème au problème (P.C) pour $\mathbb{L}$ où les $\lambda_j(t,x,\xi)$ sont égales à une quelconque des racines caractéristiques de L . Si l'on pose $p_j = \tau + \lambda_j(t,x,\xi)$ et on désigne par H_j le champ Hamiltonian de p_j , le fait que les caractéristiques de L sont involutives s'exprime par la condition:

$$\text{(IN)} \qquad H_j\, p_k = \alpha_{j,k}(\, p_j - p_k \,) \ , \ \alpha_{j,k} \in B_t^\infty(S^0) \ , \ j,k = 1,\ldots,\ell \ .$$

Au moyen de la condition (IN), on obtient une amélioration de la description donnée en (1),

$$\text{(2)} \qquad WF\, u(t_0) \subset \Gamma^r(t_0 ; WF\, g)\ , \quad t_0 > 0\ ,$$

où r est le nombre des racines caractéristiques distinctes de L .

Expliquons les définitions de $\Gamma^k(t_0 ; \tilde{K})$ et $\Gamma(t_0 ; \tilde{K})$ pour un entier $k > 0$ et un compact conique $\tilde{K}$ dans $T^*(R^n) \setminus 0$: On dit que la courbe $\{(t,x(t),\xi(t)\}$ dans $[0,t_0]\times T^*(R^n)$ obtenue en reliant k fois des courbes bicaractéristiques par rapport à λ_j est une trajectoire ε-admissible ($\varepsilon \geq 0$) à l'étape k si et seulement la différence de deux racines caractérisitiques adjacentes est majorée par $\varepsilon|\xi^j|$ à chaque jointure (t_j,x^j,ξ^j) , $j = 1,\ldots,k$. On désigne par $\Gamma^k_\varepsilon(t_0 ; \tilde{K})$ l'ensemble des extremités de toutes les trajectoires ε-admissibles à l'étape au plus k partant d'un ε-voisinage de $\tilde{K}$. On pose

$$\Gamma^k(t_0 ; \tilde{K}) = \Gamma^k_0(t_0 ; \tilde{K})\ , \quad \Gamma(t_0 ; \tilde{K}) = \bigcap_{0<\varepsilon<1} \bigcup_{k=1}^{\infty} \Gamma^k_\varepsilon(t_0 ; \tilde{K}).$$

Revenons à l'étude de la propagation du front d'onde du problème (P.C). Le fait que dans l'article [1] on a obtenu l'amélioration (2) de (1) au moyen de la condition (IN) nous mène au problème suivant : Soit m entier positif. Supposons que les caractéristiques $p_j = \tau + \lambda_j(t,x,\xi)$ de $\mathbb{L}$ vérifiant la condition de nilpotence suivante :

$$\text{(N.m)} \qquad H_{j_1} H_{j_2} \cdots H_{j_m} p_{j_{m+1}} = 0 \quad (j_k = 1,2,\ldots,\ell\ ).$$

Alors, est-ce qu'on a une amélioration de (1), c'est à dire, existe-il un entier $k = k(m,\ell) > 0$ tel que

$$\text{(3)} \qquad WF\, U(t_0) \subset \Gamma^k(t_0 ; WF\, G) \quad ?$$

La condition (N.m) avec $m = 1$ est un cas particulier de la condition (IN). L'étude de ce problème a été faite par [K-T] dans le cas $m = 1$ et par Ichinose [IC] dans le cas $m = 2$ et $\ell = 2$. Dans [IC], le nombre $k(2,2)$ qui apparait dans (3) est égal à 2.

Dans l'article [2] écrit avec C. Iwasaki, on considère le cas où $m = 3$ et $\ell = 2$. En général on ne peut espérer l'inclusion (3) pour tout k fini sans la condition

$$\text{(*)} \qquad H_1^2 p_2\, H_2^2 p_1 \leq 0 \qquad \text{en } t = 0\ .$$

En effet, il existe un exemple de système hyperbolique

$$(4) \qquad \mathbb{L}_1 = \begin{bmatrix} D_t & 0 \\ 0 & D_t + D_1 + (t^2 - 4x_1 t)D_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \text{ sur } R_t \times R_x^3$$

qui ne vérifie pas (*),pour lequel (N.3) est satisfaite et tel que pour tout k on peut construire une solution de (P.C) pour $\mathbb{L}_1$ vérifiant

$$(5) \qquad WF\, U(t_0) \cap (\Gamma^k(t_0 ; WF\, G) \setminus \Gamma^{k-1}(t_0 ; WF\, G)) \neq \emptyset .$$

Comme donnée initiale G , on choisit une modification de la fonction suivante;

$$(6) \qquad f(x) = \sum_k^{\infty} k^{-2} \phi(kx) \exp i k^2 x \cdot n \quad (n \neq 0 , \phi \in C_0^{\infty} , \hat{\phi} \geq 0 , \hat{\phi}(0) = 1).$$

La fonction f a été donnée par Hörmander et on a $WF\, f = \{(0,cn) ; c > 0 \}$. De plus, dans l'article [2] on démontre l'inclusion (3) avec k = 3 en supposant des conditions un peu fortes que (*). Dans la preuve de cette inclusion, la formule de Campbell-Hausdorff joue un rôle important.

Une autre modification de (6) est employée dans l'article [3]. On y montre la propagation précise du front d'onde le long de d'une trajectoire 0-admissible pour une classe de systèmes différentiels contenant des exemples vérifiant la condition (IN) ou (N.m). Dans l'aricle [3], on se restreint au cas où la multiplicité des caractéristiques est constante le long de la trajectoire 0-admissible.

Dans l'article [4] écrit avec C. Iwasaki, la méthode developpée dans [2] pour montrer (5) est généralisée à une certaine classe de systèmes différentiels à caractéristiques de multiplicité variable le long de la trajectoire 0-admissible. Comme application de cette étude on donne un exemple de système hyperbolique sur $R_t \times R_x^1$ pour lequel il existe une donnée initiale G telle que

$$(7) \qquad \text{supp sing } G = \{0\} , \quad \text{supp sing } U(t_0) = [-1,1] , \quad t_0 > 0 .$$

Pour une équation hyperbolique d'ordre 2, Ichinose et Kumano-go [I-K] ont montré un résultat analogue à (7). Cependant, on ne sait pas si le problème de Cauchy pour l'équation de [I-K] est bien posé jusqu'à $t = t_0$.

Les études dans [3] et [4] montrent que l'inclusion (2) et celles du type (3) obtenues dans [IC] et [2] sont optimales en général. En fin, on considère la question suivante:

(8) *Est-ce que l'inclusion (1) est optimale ?*

Remarquons que pour l'exemple $\mathbb{L}_1$ donné en (4) il existe un compact $\tilde{K}$ dans $T^*(R^n)$ tel que

$$\Gamma(t_0 ; \tilde{K}) \setminus \overline{\left(\bigcup_k^{\infty} \Gamma^k(t_0 ; \tilde{K}) \right)} \neq \emptyset \tag{9}$$

Dans l'article [5] on répond affirmativement à (8) si l'on remplace le front d'onde C^∞ dans (1) par le front d'onde θ-Gevrey qui a été défini par Hörmander. Pour un autre exemple $\mathbb{L}_2$ ayant la propriété (9) on voit qu'il existe une donnée initiale G de (P.C) telle que pour tout $2 \leq \theta < 3$ on ait

$$WF_\theta G = \tilde{K} \quad , \quad WF_\theta U(t_0) \ni \delta_\infty \quad , \tag{10}$$

où δ_∞ est un point de l'ensemble du membre de gauche de (9). Dans la preuve, on utilise l'étude de [4] et aussi le fait que $\mathbb{L}_2$ vérifie la condition (N.5).

Dans l'article [7] de la deuxième partie, on considère une classe d'opérateurs différentiels d'ordre supérieur ou égal à 2 , présentant une dégénérscence d'ordre infini (Ex. $(-\Delta_x)^\ell + g(x)(-\Delta_y)^m$, ℓ, $m > 0$ entiers et $g \in C^\infty$, $g > 0$ hors de 0 , $D^\alpha g(0) = 0$ pour tout α). On montre l'hypoellipticité de ces opérateurs en raffinant la condition suffisante d'Oleinik-Radkevich et Fediĭ.

Dans l'article [7], on étudie aussi l'hypoellipticité d'une classe d'opérateurs pseudo-différentiels à caractéristiques multiples d'ordre impair (Ex. $D_1^5 + D_2^4 + D_3^4 + i(x_1^2 x_2 (D_2^4 + D_3^4) + x_1^3 D_2^2 (D_2^2 + D_3^2))$ dans R^3 , $i = \sqrt{-1}$). Nos preuves sont inspirées de celles de Hörmander et Egorov concernant les opérateurs sous-elliptiques. L'hypoellipticité et la résolubilité pour des opérateurs pseudo-différentiels à caractéristiques doubles (Ex. $D_1^2 + (1+ix_1^k)D_2$, $k > 0$ entier) vérifiant des conditions analogues à celles de Nirenberg-Treves pour les opérateurs de type principal, ont déjà été beaucoup étudiées (Rubinstein, Wenston, Menikoff et Popivanov [R-W-M-P]). Remarquons qu'au moyen de la méthode de la décomposition et de la transformation canonique inhomogène, on peut réduire micro-localement tous les opérateurs de [R-W-M-P] à l'opérateur $D_1 + ix_1^k D_2$, qui avait été étudié par Mizohata. Par contre, l'opérateur étudié dans l'article [7] ne peut être réduit partout micro-localement au type Mizohata. En certains points, il faut le réduire au type Egorov : $D_1 + i(x_1^s D_2 + x_1^a x_2^b |D|)$, où s, a, b sont entiers positifs et $|D|$ est la racine carrée de $-\Delta_x$.

LISTE DES TRAVAUX CONSTITUANT LA THESE

1ère partie

[1] Fundamental solution for a hyperbolic equation with involutive characteristics of variable multiplicity,
Comm. in P.D.E., 4 (1979), 609-643.

[2] Avec C. Iwasaki. Propagation of singularities of solutions for a hyperbolic system with nilpotent characteristics, I,
Comm. in P.D.E., 7 (1982), 743-794.

[3] On the propagation of the wave front set of a solution for a hyperbolic system,
Math. Japonica, 27 (1982), 501-508.

[4] Avec C. Iwasaki. Propagation précise des singularités pour des systèmes hyperboliques à caractéristiques de multiplicité variable.

[5] Propagation du front d'onde Gevrey pour un exemple de système hyperbolique.

2ème partie

[6] On the hypoellipticity for infinitely degenerate semi-elliptic operators,
J. Math. Soc. Japan 30 (1978), 327-358.

[7] On hypoelliptic operators with multiple characteristics of odd order,
Osaka J. Math., 20 (1983), à paraître.

COMM. IN PARTIAL DIFFERENTIAL EQUATIONS, 4(6), 609-643 (1979)

FUNDAMENTAL SOLUTION FOR A HYPERBOLIC EQUATION WITH INVOLUTIVE CHARACTERISTICS OF VARIABLE MULTIPLICITY

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§0. Introduction. In this paper we consider a single hyperbolic operator with characteristics of variable multiplicity which has the form

$$(0.1)\qquad L = \partial_1 \cdots \partial_m + \sum_{\substack{j_1 < \cdots < j_k \\ k < m}} b_{j_1,\ldots,j_k}\, \partial_{j_1} \cdots \partial_{j_k} + c \qquad \text{on } [0,T]\times R^n ,$$

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where $\partial_j = D_t + \lambda_j(t,x,D_x)$, $j = 1,\dots,m$, for real valued $\lambda_j(t,x,\xi)$ ($\in \mathcal{B}^\infty([0,T];S^1)$), and $b_{j_1,\dots,j_k} = b_{j_1,\dots,j_k}(t,x,D_x)$, $c = c(t,x,D_x)$ belong to $\mathcal{B}^\infty([0,T];\mathcal{S}^0)$. Here $\lambda_j(t,x,\xi) \in \mathcal{B}^\infty([0,T];S^1)$ means that $\lambda_j(t,x,\xi)$ for a fixed $t \in [0,T]$ belongs to the space of symbols S^1 and $\lambda_j(t)$ is infinitely differentiable with respect to t in the topology of S^1 . We assume that the characteristics of L are involutive, in the sense that any pair of ∂_j and ∂_k satisfies

$$(0.2) \qquad [\partial_j,\partial_k] \equiv \partial_j\partial_k - \partial_k\partial_j = \alpha_{j,k}\partial_j + \beta_{j,k}\partial_k + \gamma_{j,k}$$

for some $\alpha_{j,k}(t,x,D_x)$, $\beta_{j,k}(t,x,D_x)$ and $\gamma_{j,k}(t,x,D_x) \in \mathcal{B}^\infty([0,T];\mathcal{S}^0)$. For the sake of simplicity, we also assume that

$$(0.3) \qquad \lambda_j(t,x,\delta\xi) = \delta\lambda_j(t,x,\xi) \quad (\ |\xi| \geqq M\ ,\ \delta \geqq 1\ )$$

for a constant $M > 0$.

We consider the Cauchy problem

$$(0.4) \qquad \begin{cases} L(t,x,D_t,D_x)u = f(t,x) & \text{on } [0,T]\times R^n \\ D_t^j u(0,x) = g_j(x) \quad , \quad j = 0,\dots,m-1 \ . \end{cases}$$

We show that the problem (0.4) can be reduced to the first order hyperbolic system, with a diagonal principal part, of the form

$$(0.5) \qquad \begin{cases} \mathbb{L}U = D_tU + \mathcal{B}U + BU = F(t,x) \\ U(0,x) = G(x) \qquad (\text{see } (2.1)) \end{cases}$$

for an unknown function $U = {}^t(u_0, u_{(1)}, u_{(2)}, \dots, u_{(1,2)}, u_{(2,1)}, \dots)$ with $1 + \sum_{j=1}^{m-1} m!/j!$ components (which means that the problem (0.4) is diagonalizable!). We construct the fundamental solution $\mathbb{E}(t,s)$ for the problem (0.5). Then, using the fundamental solution $\mathbb{E}(t,s)$, we obtain the representation formula for the solution u of (0.4), and we investigate the propagation of singularities for the solution u.

In case of characteristics of constant multiplicity, Chazarain [1][2] has constructed the asymtotic solution for a hyperbolic equation on the so-called "Levi condition" (c.f., Mizohata-Ohya [10][11]) and investigated the propagation of sigularities. We note that the operators with "Levi condition" always can be written in the form (0.1), and ∂_j ($j = 1,\dots,m$) of such operators satisfy (0.2) since $\sigma_0([\partial_j, \partial_k]) = \alpha_{j,k}(\lambda_j - \lambda_k)$ for some $\alpha_{j,k}(t,x,\xi) \in \mathcal{B}^\infty([0,T];S^0)$ ($\sigma_0(p)$ denotes the principal symbol of p). As an operator with characteristics of variable multiplicity, Zeman[14] (see also [13]) considered the operator of the form (0.1) satisfying (0.2) and proved the well-posedness of the Cauchy problem (0.4) by using energy estimates. The equivalence of Cauchy problems (0.4) and (0.5) (Theorem 1) gives this result since the Cauchy problem for a hyperbolic system with a diagonal principal part is well-posed.

The plan of the present paper is as follows: In §1 we reduce the Cauchy problem (0.4) to a diagonal hyperbolic system (0.5) of first order (see Theorem 1). In §2 we construct the fundamental

solution $E(t,s)$ for the system (0.5), in the same way as in Kumano-go-Taniguchi [8], and give the simplified form of the fundamental solution $E(t,s)$ if the Poisson brackets $\{ \tau + \lambda_j , \tau + \lambda_k \}$ for ∂_j ($j = 1,\dots,m$) of (0.1) satisfy

$$\{ \tau + \lambda_j , \tau + \lambda_k \} = a_{j,k}(t)(\lambda_j - \lambda_k) \tag{0.6}$$

for some $a_{j,k}(t) \in C^{\infty}([0,T])$ independent of (x,ξ), where τ denotes the dual variable of t. In §3 we investigate the propagation of singularities for the solution u of (0.4), by using the discussion in [8], and we get a sharp form of Theorem 3.4 of [8] under the assumption (0.2).

§1. Reduction to Diagonal First Order System.

First we introduce some notations before the reduction to a first order system. Let Π_k denote the set which consists of k-permutations $(j_1,\dots,j_k)$ of elements of $\{1,\dots,m\}$ and let $\tilde{\Pi}_k$ be a subset of Π_k whose elements satisfy $j_1 < \cdots < j_k$. Set $\Pi = \bigcup_{k=0}^{m-1} \Pi_k$ and $\tilde{\Pi} = \bigcup_{k=0}^{m-1} \tilde{\Pi}_k$ where $\Pi_0 = \tilde{\Pi}_0 = \{0\}$. For a permutation $J = (j_1,\dots,j_k) \in \Pi_k$ and $\partial_1 , \dots , \partial_m$ in (0.1) we denote $\partial_{j_1} \cdots \partial_{j_k}$ by ∂^J (, where $\partial^0 = 1$). Let $|J|$ denote the length k of $J \in \Pi_k$. Using these notations we can write (0.4) simply in the form

$$\begin{cases} Lu \equiv \partial_1 \cdots \partial_m u + \sum_{J \in \tilde{\Pi}} b_J \partial^J u = f(t,x) & \text{on } [0,T]\times R^n \\ D_t^j u(0,x) = g_j(x) \quad , \quad 0 \le j \le m-1 \quad . \end{cases} \tag{1.1}$$

Furthermore, from the assumption (0.2) we have for any $J_o \in \Pi_m$

$$\partial^{J_o} = \partial_1 \cdots \partial_m + \sum_{J \in \Pi} a_J^{J_o} \partial^J \tag{1.2}$$

for some $a_J^{J_o}(t,x,D_x) \in \mathcal{B}^\infty([0,T];\mathcal{B}^0)$. Of course, the representation of the right side is not unique but we fix one for each $J_o \in \Pi_m$ (such that $a_J^{J_o} = 0$ for $J_o = (1,\ldots,m)$) in what follows.

Now for the equation (1.1) we make a diagonal hyperbolic system of first order for an unknown vector valued function

$$U = {}^t(u_0 , u_{(1)}, u_{(2)},\ldots, u_{(1,2)}, u_{(2,1)},\ldots, u_J ,\ldots) \quad (J \in \Pi)$$

as follows:

i) In case $|J| \leqq m-2$ ($J = (j_1,\ldots,j_k)$, $k \leqq m-2$), for the largest number ℓ ($1 \leqq \ell \leqq m$) such that $\ell \notin \{j_1,\ldots,j_k\}$ we set

$$\partial_\ell u_J = u_{J'} \quad , \quad J' = (\ell,j_1,\ldots,j_k) \in \Pi_{k+1} \tag{1.3}$$

ii) When $|J| = m-1$ ($J = (j_1,\ldots,j_{m-1})$), for $\ell \notin \{j_1,\ldots,j_{m-1}\}$ we set

$$\partial_\ell u_J = f - \sum_{J' \in \tilde{\Pi}} b_{J'} u_{J'} + \sum_{J' \in \Pi} a_{J'}^{J_o} u_{J'} \quad , \tag{1.4}$$

where $J_o = (\ell,j_1,\ldots,j_{m-1})$ and $a_{J'}^{J_o}$ are the same as in (1.2). Noting that ∂^J ($J = (j_1,\ldots,j_k) \in \Pi_k$) can be written in the form

$$\partial^J = D_t^k + \sum_{j=0}^{k-1} C_j^J(t,x,D_x) D_t^j \tag{1.5}$$

for some $C_j^J(t,x,D_x) \in \mathcal{B}^\infty([0,T];\mathcal{B}^{k-j})$, we give the initial condition for U

$$(1.6)\quad \begin{cases} u_0(0,x) = g_0(x) \\ u_J(0,x) = g_{|J|}(x) + \sum_{j=0}^{|J|-1} C_j^J(0,x,D_x)g_j(x) \quad , \quad J \in \Pi \ . \end{cases}$$

It is easy to see that if $u \in \mathcal{B}_t^m(H_{-\infty})$ is a solution of (1.1), then $U = {}^t(u, \partial_1 u, \partial_2 u, \ldots, \partial_1\partial_2 u, \partial_2\partial_1 u, \ldots, \partial^J u, \ldots) \in \mathcal{B}_t^1(H_{-\infty})$ becomes a solution of the system composed of (1.3), (1.4) and (1.6) (, where $H_{-\infty} = \bigcup_\sigma H_\sigma$ for the Sobolev space H_σ). Conversely we have

Theorem 1. If $U \in \mathcal{B}_t^1(H_{-\infty})$ is a solution of the system, then the first component u_0 of U is a solution of (1.1) and belongs to $\mathcal{B}_t^m(H_{-\infty})$.

Proof. From (1.3) it follows that

$$(1.7)\quad \begin{aligned} & u_{(m)} = \partial_m u_0 \quad , \quad u_{(m-1,m)} = \partial_{m-1} u_{(m)} = \partial_{m-1}\partial_m u_0 \ , \\ & \qquad\qquad \ldots\ldots \\ & u_{(2,\ldots,m)} = \partial_2 u_{(3,\ldots,m)} = \partial_2\partial_3\cdots\partial_m u_0 \ . \end{aligned}$$

In view of this and (1.4) we have

$$(1.8)\quad \partial_1 u_{(2,\ldots,m)} = \partial_1\partial_2\cdots\partial_m u_0 = f - \sum_{J' \in \tilde{\Pi}} b_{J'} u_{J'} \ .$$

If we show $u_J = \partial^J u_0$, we see that u_0 satisfy the first equation of (1.1). To derive this fact, we set $w_J = u_J - \partial^J u_0$ ($J \in \Pi$) and consider for $W = {}^t(w_0 , w_{(1)}, w_{(2)}, \ldots, w_J , \ldots)$ the following system: If $|J| \leq m-2$ ($J = (j_1,\ldots,j_k)$), then from (1.3) we get

$$(1.9)\quad \partial_\ell w_J = w_{J'} \quad , \quad J' = (\ell, j_1, \ldots, j_k) \ .$$

When $|J| = m-1$ ($J = (j_1,\dots,j_{m-1})$), from (1.4) we obtain

$$\begin{aligned}\partial_\ell w_J &= f - \sum_{J'\in\tilde{\Pi}} b_{J'}u_{J'} + \sum_{J'\in\Pi} a_{J'}u_{J'} - \partial^{J_o}u_0 \\ &= f - \sum_{J'\in\tilde{\Pi}} b_{J'}u_{J'} + \sum_{J'\in\Pi} a_{J'}^{J_o}w_{J'} \\ &\quad + \sum_{J'\in\Pi} a_{J'}\partial^{J'}u_0 - \partial^{J_o}u_0 , \quad J_o = (\ell,j_1,\dots,j_{m-1}).\end{aligned}$$

Using (1.8) and (1.2),

$$\partial_\ell w_J = \partial_1\cdots\partial_m u_0 + \sum_{J'\in\Pi} a_{J'}w_{J'} - \partial_1\cdots\partial_m u_0$$

Therefore,

$$\partial_\ell w_J = \sum_{J'\in\Pi} a_{J'}^{J_o}w_{J'} , \quad J_o = (\ell,j_1,\dots,j_{m-1}) . \tag{1.10}$$

Since this and (1.9) make a homogeneous diagonal hyperbolic system for W, we have $W(t,x) = 0$ if we show $W(0,x) = 0$. If we rewrite (1.7) by using (1.5), then from (1.6) we can show inductively

$$D_t^j u_0(0,x) = g_j(x) \qquad (j = 0,\dots,m-1).$$

Moreover, this and (1.5) lead us to

$$\partial^J u_0(0,x) = g_{|J|}(x) + \sum_{j=0}^{|J|-1} C_j^J(0)g_j(x) \qquad (J\in\Pi).$$

From this and (1.6) we obtain $w_J(0,x) = 0$. In view of $\partial_J u_0 = u_J$ ($J\in\Pi$) and (1.5), we can easily get the last statement.

Q.E.D.

Remark 1. The proof of theorem is based on the uniqueness of the solution for the Cauchy problem (1.9)-(1.10) at $t = 0$.

Hence, the theorem is true in case of λ_j complex valued, if λ_j ($j = 1,\dots,m$) satisfy the conditions treated in Zeman [13].

Remark 2. Recently, Uryu [12] also has treated the Cauchy problem for the operator of the form (0.1) by reducing to a diagonal system in some different stiuation.

§2. Construction of Fundamental Solution.

We consider the diagonal first order hyperbolic system, determined by (1.3), (1.4) and (1.6), which has the form

$$(2.1)\quad \begin{cases} \mathbb{L}U \equiv D_t U + \mathcal{D}U + BU = F(t,x) & \text{on } [0,T]\times R^n \\ U(0,x) = G(x) \quad , \quad x \in R^n \quad , \end{cases}$$

where $\mathcal{D}$ and B are $M\times M$ matrices ($M = 1 + \sum_{j=1}^{m-1} m!/j!$),

$$\mathcal{D} = \begin{pmatrix} \lambda_m & & 0 \\ & \ddots & \\ & & \lambda_\ell \\ & \ddots & \\ 0 & & \ddots \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 & & \ddots & 0 \\ & \ddots & \ddots & & \\ & & \ddots & & -1 \\ -a^J_{J'}+b_{J'} & & & & \end{pmatrix} \quad \begin{pmatrix} J \in \Pi \\ J' \in \Pi_{m-1} \end{pmatrix}$$

$$F = {}^t(0,\dots,0,f,\dots,f)$$

$$G = {}^t\Big(g_0 , \dots , g_{|J|} + \sum_{j=0}^{|J|-1} C^J_j(0) g_j , \dots \Big) .$$

We say that a C^∞ phase function $\phi(x,\xi)$ in $R^n_x \times R^n_\xi$ belongs to $\mathcal{P}(\tau)$ ($0 \le \tau < 1$), if $J(x,\xi) \equiv \phi(x,\xi) - x\cdot\xi$ is of class S^1 and satisfies

$$\| J \|_0 \equiv \max_{|\alpha+\beta| \le 2} \sup_{x,\xi} \{ |\partial_\xi \partial_x J(x,\xi)| / \langle\xi\rangle^{1-|\alpha|} \} < \tau .$$

Let $\phi_j(t,s) = \phi_j(t,s;x,\xi)$ ($j = 1,\dots,m$) be the solutions of eiconal equations

$$(2.2)\quad \begin{cases} \partial_t \phi_j + \lambda_j(t,x,\nabla_x \phi_j) = 0 \quad (0 \leq s \leq t \leq T) \\ \phi_j \big|_{t=s} = x\cdot\xi . \end{cases}$$

If we choose a sufficiently small $T > 0$, then phase functions $\phi_j(t,s)$ exist and belong to $\mathcal{P}(c_0(t-s))$ for a constant $c_0 > 0$ (See [5]). Furthermore, for any sequence of phase functions $\phi_{j_1}(t,t_1), \phi_{j_2}(t_1,t_2),\dots, \phi_{j_{\nu+1}}(t_\nu,s)$ ($0 \leq s \leq t_\nu \leq \cdots \leq t_1 \leq t \leq T$) we can define the multi-product $\phi_{j_1,\dots,j_{\nu+1}} = \phi_{j_1} \# \phi_{j_2} \# \cdots \# \phi_{j_{\nu+1}}$ of class $\mathcal{P}(c_1(t-s))$ for another constant $c_1 > 0$ by

$$(2.3)\quad \begin{aligned} &\phi_{j_1,\dots,j_{\nu+1}}(t,t_1,\dots,t_\nu,s;x^0,\xi^{\nu+1}) \\ &= \sum_{k=1}^{\nu} (\phi_{j_k}(t_{k-1},t_k;X_\nu^{k-1},\Xi_\nu^k) - X_\nu^k \cdot \Xi_\nu^k) + \phi_{j_{\nu+1}}(t_\nu,s;X_\nu^\nu,\xi^{\nu+1}) \\ &\qquad (t_0 = t \ , \ X_\nu^0 = x^0), \end{aligned}$$

where $\{X_\nu^k,\Xi_\nu^k\}_{k=1}^{\nu}(t,t_1,\dots,t_\nu,s;x^0,\xi^{\nu+1})$ is defined as the solution of the equation

$$(2.4)\quad \begin{cases} x^k = \nabla_\xi \phi_{j_k}(t_{k-1},t_k;x^{k-1},\xi^k) \\ \xi^k = \nabla_x \phi_{j_{k+1}}(t_k,t_{k+1};x^k,\xi^{k+1}) \quad , \quad k = 1,\dots,\nu \end{cases}$$

(See [7]). We fix such a small $T > 0$ in what follows. It should be noted that the fixed $T > 0$ can be chosen to be independent of ν .

Let $J_\nu = (j_1,\dots, j_{\nu+1})$ $(\nu = 0,\dots)$ be a $(\nu+1)$-repeated-permutation of elements of $\{ 1,\dots,m \}$ and let $\mathcal{J}_{\nu+1}$ $(\nu = 0,\dots)$ denote the set of $(\nu+1)$-repeated-permutations J_ν. Then, for the brevity we often use a notation J_ν instead of a suffix $j_1,\dots,j_{\nu+1}$, such as $\Phi_{J_\nu} = \Phi_{j_1,\dots,j_{\nu+1}}$.

From the definition of Φ_{J_ν} we have the following

Proposition. $\Phi = \Phi_{J_\nu}(t_0,t_1,\dots,t_\nu,t_{\nu+1})$ satisfies

1°.

$$(2.5)\quad \begin{cases} \partial_{t_0}\Phi = -\lambda_{j_1}(t_0,x^0,\ \nabla_{x^0}\Phi) \\ \partial_{t_k}\Phi = \lambda_{j_k}(t_k,X_\nu^k,\Xi_\nu^k) - \lambda_{j_{k+1}}(t_k,X_\nu^k,\Xi_\nu^k) \qquad (k = 1,\dots,\nu) \\ \partial_{t_{\nu+1}}\Phi = \lambda_{j_{\nu+1}}(t_{\nu+1},\ \nabla_{\xi^{\nu+1}}\Phi\ ,\xi^{\nu+1}) \ . \end{cases}$$

2°. If $t_{k-1} = t_k$ for some k, we have

$$(2.6)\quad \Phi_{J_\nu}(t_0,t_1,\dots,t_{k-1},t_k,t_{k+1},\dots,t_{\nu+1};x^0,\xi^{\nu+1}) = \Phi_{\tilde{J}_{\nu-1}}(t_0,t_1,\dots,t_{k-1},t_{k+1},\dots,t_{\nu+1};x^0,\xi^{\nu+1}) ,$$

where $\tilde{J}_{\nu-1} = (j_1,j_2,\dots,j_{k-1},j_{k+1},\dots,j_{\nu+1})$.

3°. If $j_k = j_{k+1}$ (therefore $\phi_{j_k} = \phi_{j_{k+1}}$) for some k, then we have (2.6).

Proof is omitted. See Theorem 2.3 of [7].

Now we construct the fundamental solution $\mathbb{E}(t,s)$ for $\mathbb{L}$ of (2.1) in the same way as in Theorem 3.1 of [8]. Set

$$I_\phi(t,s) = \begin{pmatrix} I_{\phi_m} & & 0 \\ & \ddots & \\ & & I_{\phi_\ell} \\ 0 & & & \ddots \end{pmatrix}$$

where I_{ϕ_ℓ} are Fourier integral operators with phase functions $\phi_\ell(t,s;x,\xi)$ and symbol 1, that is,

$$I_{\phi_\ell} u(t,s,x) = (2\pi)^{-n} \int e^{i\phi_\ell(t,s;x,\xi)} \hat{u}(\xi) d\xi \ , \quad u \in \mathcal{A}.$$

Here $\hat{u}(\xi)$ denotes the Fourier transform of u. Then, we have by [5]

$$\mathbb{L} I_\phi(t,s) = R_\phi(t,s) \tag{2.7}$$

where $R_\phi(t,s) = \sum_{j=1}^{m} R_{j,\phi_j}(t,s)$ is a matrix of Fourier integral operators with phase functions $\phi_j(t,s)$ and symbols $r_j(t,s;x,\xi)$ of class $\mathcal{B}^\infty(\Delta, S^0)$ ($\Delta = \{ 0 \leqq s \leqq t \leqq T \}$). Noting that $R_\phi(t,s)$ is the continuous operator from the Sobolev space H_σ into itself for any σ, we can construct the fundamental solution $\mathbb{E}(t,s)$ for $\mathbb{L}$, as the continuous operator from H_σ into itself, in the form:

$$\mathbb{E}(t,s) = I_\phi(t,s) + \int_s^t I_\phi(t,\theta) \sum_{\nu=1}^{\infty} W_\nu(\theta,s) d\theta \ , \tag{2.8}$$

where $\{W_\nu(t,s)\}_{\nu=1}^{\infty}$ are defined by

$$W_1(t,s) = -iR_\phi(t,s)$$

$$W_{\nu+1}(t,s) = \int_s^t W_1(t,\theta) W_\nu(\theta,s) d\theta \qquad (\nu = 1,2,\dots)$$

$$\left(= \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} W_1(t,t_1) W_1(t_1,t_2) \cdots W_1(t_\nu,s) dt_1 dt_2 \cdots dt_\nu, \right.$$

$$\left. t_0 = t \right).$$

Indeed, if we set $C_1 = \max_{0 \leq s \leq t \leq T} \| R_\phi(t,s) \|_{H_\sigma \to H_\sigma}$ then we have $\| W_{\nu+1}(t,s) \|_{H_\sigma \to H_\sigma} \leq C_1^\nu (t-s)^\nu / \nu!$ inductively. Hence, the infinite series of (2.8) converges in the space of continuous operators from H_σ into itself. Furthermore, we show that $E(t,s)$ can be represented in the form of infinite series of Fourier integral operators. Note that

$$(2.9) \quad W_1(t,t_1)W_1(t_1,t_2)\cdots W_1(t_\nu,s) = \sum_{J_\nu \in \mathcal{J}_{\nu+1}} (-i)^{\nu+1} R_{j_1,\phi_{j_1}}(t,t_1)\cdots R_{j_{\nu+1},\phi_{j_{\nu+1}}}(t_\nu,s) .$$

From the fundamental theorem of the multi-product of Fourier integral operators (see Theorem 2.5 of [8]) we can find all terms on the right side of (2.9) Fourier integral operators $W_{J_\nu;\Phi_{J_\nu}}(t,t_1,\ldots,t_\nu,s)$ with phase functions $\Phi_{J_\nu}(t,t_1,\ldots,t_\nu,s)$ and symbols $w_{J_\nu}(t,t_1,\ldots,t_\nu,s)$ of class $\mathcal{B}^\infty(\Delta_\nu;S^0)$ ($\Delta_\nu = \{0 \leq s \leq t_\nu \leq \cdots \leq t_1 \leq t \leq T$), as follows: For any multi-index $\alpha = (\alpha_0,\alpha_1,\ldots,\alpha_{\nu+1})$ satisfying $|\alpha| = j$ and any integer ℓ we have semi-norm estimates

$$(2.10) \quad |\partial_{\underline{t}}^\alpha w_{J_\nu}(t,t_1,\ldots,t_\nu,s)|^{(0)}_\ell \leq C_{j,\ell}^\nu$$

$$(\quad \partial_{\underline{t}}^\alpha \equiv \partial_t^{\alpha_0} \partial_{t_1}^{\alpha_1} \cdots \partial_s^{\alpha_{\nu+1}} , \quad |\alpha| \equiv \Sigma \alpha_j \quad)$$

for a constant $C_{j,\ell}$ independent of ν , and moreover for any α ($|\alpha| = j$), real σ and integer $N > 0$ we have

$$(2.11)\quad \| \partial_t^{\ell} \{ R_{j_1,\phi_{j_1}}(t,t_1) R_{j_2,\phi_{j_2}}(t_1,t_2) \cdots R_{j_{\nu+1},\phi_{j_{\nu+1}}}(t_\nu,s) - W_{J_\nu;\phi_{J_\nu}}(t,t_1,\ldots,t_\nu,s) \} \|_{H_\sigma \to H_{\sigma+N}} \leq C_{j,\sigma,N}^{\nu}$$

for a constant $C_{j,\sigma,N}$ independent of ν . Set

$$(2.12)\quad W_{\nu+1,\phi_{\nu+1}}(t,t_1,\ldots,t_\nu,s) = \sum_{J_\nu \in \mathcal{J}_{\nu+1}} (-i)^{\nu+1} W_{J_\nu;\phi_{J_\nu}}(t,t_1,\ldots,t_\nu,s)$$

and set

$$(2.13)\quad W_{-\infty}(t,s) = \int_s^t I_\phi(t,\theta) \left[\sum_{\nu=1}^{\infty} \int_s^\theta \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} \{ W_1(\theta,t_1) W_1(t_1,t_2) \cdots W_1(t_\nu,s) - W_{\nu+1,\phi_{\nu+1}}(\theta,t_1,\ldots,t_\nu,s) \} dt_1 dt_2 \cdots dt_\nu \right] d\theta \qquad (t_0 = \theta).$$

Then we have

Theorem 2. There exists the fundamental solution $\mathbb{E}(t,s)$ for $\mathbb{L}$ in the sense of

$$(2.14)\quad \mathbb{L}\mathbb{E}(t,s) = 0 \quad , \quad \mathbb{E}(s,s) = I \quad , \quad (0 \leq s \leq t \leq T),$$

which can be represented in the form:

$$(2.15)\quad \mathbb{E}(t,s) = I_\phi(t,s) + \int_s^t I_\phi(t,\theta) \{ W_1(\theta,s) + \sum_{\nu=1}^{\infty} \int_s^\theta \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} W_{\nu+1;\phi_{\nu+1}}(\theta,t_1,\ldots,t_\nu,s) dt_1 dt_2 \cdots dt_\nu \} d\theta + W_{-\infty}(t,s)$$

$$(t_0 = \theta , W_{-\infty}(t,s) \in \mathcal{B}^{\infty}(\Delta;\mathcal{S}^{-\infty}), \Delta = \{0 \le s \le t \le T\}),$$

where $W_{\nu+1;\Phi_{\nu+1}}$ and $W_{-\infty}(t,s)$ are defined by (2.12) and (2.13), respectively.

For the proof we may only prove that $W_{-\infty}(t,s) \in \mathcal{B}^{\infty}(\Delta;\mathcal{S}^{-\infty})$, which is given in the proof of Theorem 3.1 of [8].

Under the following new assumption (2.16) we can simplify the representation of the fundamental solution, and get a slightly extended form of Theorem 3.3 of [8] which is obtained as the generalization of the results by Ludwig-Granoff [9] and Hata [3].

Theorem 3. Assume that the Poisson brackets $\{\tau + \lambda_j , \tau + \lambda_k\}$ for $\partial_j = D_t + \lambda_j(t,x,D_x)$ ($j = 1,\dots,m$) satisfy

$$(2.16) \quad \{\tau + \lambda_j , \tau + \lambda_k\}$$

$$\equiv \partial_t\lambda_j - \partial_t\lambda_k + \nabla_x\lambda_j\cdot\nabla_\xi\lambda_k - \nabla_x\lambda_k\cdot\nabla_\xi\lambda_j$$

$$= a_{j,k}(t) (\lambda_j - \lambda_k) \qquad (j,k = 1,\dots,m)$$

for some $a_{j,k}(t) \in C^{\infty}([0,T])$. Then we can represent $\mathbb{E}(t,s)$ in the form

$$(2.17) \quad \mathbb{E}(t,s) = \sum_{j=1}^{m} \tilde{W}_{j,\phi_j}(t,s)$$

$$+ \sum_{k=1}^{m-1} \sum_{J_k \in \tilde{\Pi}_{k+1}} \int_s^t \cdots \int_s^{t_{k-1}} \tilde{W}_{J_k;\Phi_{J_k}}(t,t_1,\dots,$$

$$t_k,s)dt_1dt_2\cdots dt_k + W_{-\infty}(t,s)$$

$$(t_0 = t , W_{-\infty}(t,s) \in \mathcal{B}^{\infty}(\Delta;\mathcal{S}^{-\infty})).$$

Here $W_{J_k;\Phi_{J_k}}(t,t_1,\dots,t_k,s)$ are matrices of Fourier integral operators with phase functions $\Phi_{J_k}(t,t_1,\dots,t_k,s)$ and symbols $w_{J_k}(t,t_1,\dots,t_k,s)$ of class $\mathcal{B}^{\infty}(\Delta_k;S^0)$ ($\Delta_k = \{ 0 \leq s \leq t_k \leq \cdots \leq t_1 \leq t \leq T \}$).

Remark 1. It is easy to see that the assumption (0.2) is equivalent to the fact that

$$\{ \tau + \lambda_j , \tau + \lambda_k \} = \tilde{a}_{j,k}(t,x,\xi)(\lambda_j - \lambda_k) \qquad (2.18)$$

$$(j,k = 1,\dots,m)$$

for some $\tilde{a}_{j,k}(t,x,\xi) \in \mathcal{B}^{\infty}([0,T];S^0)$. The condition (2.16) means $\tilde{a}_{j,k}$ is independent of (x,ξ).

Remark 2. We can give the following examples satisfying (2.16):

i) $\lambda_j(t,x,\xi) = b(t)(x_1 + c_j x_2 + d_j x_3)\xi_1$ ($j = 1,\dots,m$).

ii) $\lambda_j(t,x,\xi) = b(t)x_1(\xi_1 + c_j\xi_2 + d_j\xi_3)$ ($j = 1,\dots,m$). Here $b(t) \neq 0$ and c_j , d_j are real numbers. Indeed, the Poisson brackets of these are

$$\{ \tau + \lambda_j , \tau + \lambda_k \} = (b - \partial_t b / b)(\lambda_j - \lambda_k).$$

We prepare the following

Lemmma 1. Let $\phi_j(t,s;x,\xi)$ be the solution of (2.2). For a permutation $J_\nu = (j_1,\dots,j_{\nu+1})$ we set

$$\Phi_{J_\nu}(t,t_1,\dots,t_\nu,s) = \phi_{j_1}(t,t_1)\#\phi_{j_2}(t_1,t_2)\#\cdots\#\phi_{j_{\nu+1}}(t_\nu,s).$$

Assume that λ_j ($j = 1,\dots,m$) satisfy (2.16). Let $\mu_{j,k}(t)$

be the solutions of differential equations

$$(2.19)\quad d\mu_{j,k}/dt = \exp(A_{j,k}(t)), \quad dA_{j,k}/dt = a_{j,k}(t) \qquad (j,k = 1,\dots,m)$$

and set

$$(2.20)\quad \omega_{j,k}(t,t_1,s) = \mu_{j,k}^{-1}(\mu_{j,k}(t) - \mu_{j,k}(t_1) + \mu_{j,k}(s)) .$$

Then, we have

$$(2.21)\quad \Phi_{2,1}(t,t_1,s) = \Phi_{1,2}(t, \omega_{1,2}(t,t_1,s), s)$$

and moreover

$$(2.22)\quad \Phi_{\tilde{J}_\nu}(t,t_1,\dots,t_\nu,s) = \Phi_{J_\nu}(t,\dots,t_{k-1}, \omega_{j_k,j_{k+1}}(t_{k-1},t_k,t_{k+1}), t_k,\dots,s) ,$$

where $\tilde{J}_\nu = (j_1,\dots,j_{k-1},j_{k+1},j_k,j_{k+2},\dots,j_{\nu+1})$.

Remark. It is easy to check that $\omega_{j,k}(t,t_1,s)$ is independent of the initial conditions for $\mu_{j,k}$ and $A_{j,k}$ in (2.19). If $a_{j,k}(t) \equiv 0$, then $\omega_{j,k}(t,t_1,s) = t - t_1 + s$ (c.f., the corollary to Theorem 2.3 of [7]).

Proof. We may only prove (2.21) since the associated law holds in the product of phase functions (See Theorem 1.9 of [7] or Theorem 1.6 of [8]). Let (X_1^1, Ξ_1^1) be the solution of (2.4) with $\nu = 1$, and let $(y(\sigma), \eta(\sigma))$ be the bicharacteristic curve of $\lambda_1(t,x,D_x)$ which passes through the point $(x^0, \nabla_x\phi_1(t,t_1,x^0,\Xi_1^1))$

when $\sigma = t$. Then we have $y(t_1) = X_1^1$, $\eta(t_1) = \Xi_1^1$ and moreover from (2.16)

$$(2.23)\quad \frac{d}{d\sigma}\{\lambda_1(\sigma,y(\sigma),\eta(\sigma)) - \lambda_2(\sigma,y(\sigma),\eta(\sigma))\}$$

$$= \frac{d}{d\sigma}\{\tau + \lambda_1\}(\sigma,y(\sigma),\eta(\sigma)) - \frac{d}{d\sigma}\{\tau + \lambda_2\}(\sigma,y(\sigma),\eta(\sigma))$$

$$= \{\tau + \lambda_1 , \tau + \lambda_2\}(\sigma,y(\sigma),\eta(\sigma))$$

$$= a_{1,2}(\sigma)\,(\lambda_1(\sigma,y(\sigma),\eta(\sigma) - \lambda_2(\sigma,y(\sigma),\eta(\sigma)))\,.$$

Hence,

$$(2.24)\quad \lambda_1(t_1,X_1^1,\Xi_1^1) - \lambda_2(t_1,X_1^1,\Xi_1^1)$$

$$= \{\lambda_1(t,x^0,\nabla_x\phi_1(t,t_1;x^0,\Xi_1^1)) - \lambda_2(t,x^0,\nabla_x\phi_1(t,t_1;x^0,\Xi_1^1))\}$$

$$\times \exp(\,A(t_1) - A(t)\,)\,,\quad (\,A = A_{1,2}\,).$$

Noting that $\nabla_x\phi_1(t,t_1;x^0,\Xi_1^1) = \nabla_x\Phi_{1,2}(t,t_1,s;x^0,\xi^2)$, from (2.5) and (2.24) we obtain

$$(2.25)\quad \partial_{t_1}\Phi_{1,2}(t,t_1,s)$$

$$= \{\lambda_1(t,x^0,\nabla_x\Phi_{1,2}) - \lambda_2(t,x^0,\nabla_x\Phi_{1,2})\}\exp(\,A(t_1) - A(t))\,.$$

Since $a_{1,2} = a_{2,1}$, we get in the similar way

$$(2.26)\quad \partial_{t_1}\Phi_{2,1}(t,t_1,s)$$

$$= \{\lambda_2(t,x^0,\nabla_x\Phi_{2,1}) - \lambda_1(t,x^0,\nabla_x\Phi_{2,1})\}\exp(\,A(t_1) - A(t)\,)\,.$$

In view of (2.19) and (2.20) we have by (2.25)

$$\partial_{t_1}\Phi_{1,2}(t,\ \omega(t,t_1,s),\ s)$$

$$= (\partial_{t_1}\Phi_{1,2})(t,\ \omega(t,t_1,s)\ s)\ \frac{\partial\omega}{\partial t_1}(t,t_1,s)$$

$$= (\lambda_1(t,x^0,\ \nabla_x\Phi_{1,2}) - \lambda_2(t,x^0,\nabla_x\Phi_{1,2}))\ \exp(\ A(\omega) - A(t)\ )$$

$$\times \exp(-A(\omega)\ )\ (-1)\ \exp A(t_1)\quad .$$

This and (2.26) show that $\Phi_{1,2}(t,\ \omega(t,t_1,s),\ s)$ and $\Phi_{2,1}(t,t_1,s)$ satisfy the same differential equation. On the other hand from (2.6) we get

$$\Phi_{1,2}(t,\ \omega(t,s,s),\ s) = \phi_2(t,s) = \Phi_{2,1}(t,s,s)$$

So we obtain (2.20). Q.E.D.

Proof of Theorem 3. We devide the proof into four steps.

i) Let $W_{J_\nu;\Phi_{J_\nu}}(t,t_1,\ldots,t_\nu,s)$ be a Fourier integral operator with phase function $\Phi_{J_\nu}(t,t_1,\ldots,t_\nu,s)$ ($J_\nu = (j_1,\ldots,j_{\nu+1})$) and symbol $w_{J_\nu}(t,t_1,\ldots,t_\nu,s) \in \mathcal{B}^\infty(\Delta_\nu;S^0)$. Suppose that J_ν satisfy

$$(2.27)\qquad \{j_1,\ldots,j_{\nu+1}\} = \{\tilde{j}_1,\ldots,\tilde{j}_{k+1}\} \qquad (\ k+1 \leqq m\ )$$

for a $J_k = (\tilde{j}_1,\ldots,\tilde{j}_{k+1}) \in \tilde{\Pi}_{k+1}$. Then we can find a Fourier integal operator $\tilde{W}_{J_\nu;\Phi_{J_k}}(t,t_1,\ldots t_k,s)$ with phase function $\Phi_{J_k}(t,t_1,\ldots,t_k,s)$ and symbol $\tilde{w}_{J_\nu}(t,t_1,\ldots,t_k,s) \in \mathcal{B}^\infty(\Delta_k;S^0)$ such that

$$(2.28)\quad I \equiv \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} W_{J_\nu;\Phi_{J_\nu}}(t,t_1,\ldots,t_\nu,s)dt_1\cdots dt_\nu$$
$$= \int_s^t \int_s^{t_1} \cdots \int_s^{t_{k-1}} \tilde{W}_{J_\nu;\Phi_{J_k}}(t,t_1,\ldots,t_k,s)dt_1\cdots dt_k .$$

Assume that the symbol $w_{J_\nu}(t,t_1,\ldots,t_\nu,s)$ satisfy (2.10). Then, we can show that for any $\alpha = (\alpha_0,\ldots,\alpha_{k+1})$, and any integer ℓ there exists some constant $C_{\alpha,\ell}$ independent of ν such that

$$(2.29)\quad \left|\partial_t^\alpha \tilde{w}_{J_\nu}(t,t_1,\ldots,t_k,s)\right|_\ell^{(0)} \leq C_{\alpha,\ell}{}^\nu /(\nu-k)!$$
$$(\ \nu = m+1,\ldots.\ ).$$

If we admit (2.28) and (2.29), the proof is completed. Indeed from (2.15) and (2.28) we obtain (2.17) and by (2.29) we see that $w_{J_k}(t,t_1,\ldots,t_k,s) \equiv \sum_{\nu=k+1}^{\infty} \sum_{J_\nu} \tilde{w}_{J_\nu}(t,t_1,\ldots,t_k,s)$ belongs to $\mathcal{B}^\infty(\Delta_k;s^0)$ (The second Σ denotes the summation with respect to J_ν satisfying (2.27)).

ii) Set $\tilde{J}_\nu = (j_1,\ldots,j_{k-1},j_{k+1},j_k,j_{k+2},\ldots,j_{\nu+1})$ and set

$$(2.30)\quad w_{\tilde{J}_\nu}(t,t_1,\ldots,t_\nu,s)$$
$$= w_{J_\nu}(t,\ldots,t_{k-1},\ \omega(t_{k-1},t_k,t_{k+1}),t_{k+1},\ldots,t_{\nu+1})$$
$$\times \frac{\partial\omega}{\partial t_k}(t_{k-1},t_k,t_{k+1}) \quad , \qquad (\ \omega = \omega_{j_k,j_{k+1}}\ ).$$

Let $W_{\tilde{J}_\nu;\Phi_{\tilde{J}_\nu}}$ be a Fourier integral operator with phase function $\Phi_{\tilde{J}_\nu}(t,t_1,\ldots,t_\nu,s)$ and symbol $w_{\tilde{J}_\nu}(t,t_1,\ldots,t_\nu,s)$. Note that

$$(2.31)\quad I = \int_s^t dt_1 \cdots \int_s^{t_{k-2}} dt_{k-1} \int_s^{t_{k-1}} dt_{k+1} \int_s^{t_{k+1}} dt_{k+2} \cdots \int_s^{t_{\nu-1}} dt_\nu \times \int_{t_{k+1}}^{t_{k-1}} W_{J_\nu;\Phi_{J_\nu}} dt_k .$$

By (2.22) and (2.30) we get

$$\int_{t_{k+1}}^{t_{k-1}} W_{J_\nu;\Phi_{J_\nu}} dt_k = \int_{t_{k+1}}^{t_{k-1}} W_{\tilde{J}_\nu;\Phi_{\tilde{J}_\nu}} dt_k .$$

Therefore we have

$$(2.32)\quad I = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} W_{\tilde{J}_\nu;\Phi_{\tilde{J}_\nu}}(t,t_1,\ldots,t_\nu,s)dt_1\cdots dt_\nu .$$

iii) For a fixed $N \geqq 3m$ we devide $J_\nu = (j_1,\ldots,j_{\nu+1})$ into $(j_1,\ldots,j_N),(j_{N+1},\ldots,j_{2N}),\ldots,(j_{N[\nu/N]+1},\ldots,j_{\nu+1})$ and transpose each $(j_{\kappa N+1},\ldots,j_{(\kappa+1)N})$ into $(j^0_{\kappa N+1},\ldots,j^0_{(\kappa+1)N})$ satisfying $j^0_{\kappa N+1} \leqq \cdots \leqq j^0_{(\kappa+1)N}$ (, where $\kappa = 0,\ldots,[\nu/N]$ and $([\nu/N]+1)N = \nu+1$). Set $J^0_\nu = (j^0_1,\ldots,j^0_{\nu+1})$. The number of transpositions ($(j_k,j_{k+1}) \to (j_{k+1},j_k)$) to change J_ν into J^0_ν is not more than $C_{N,m}([\nu/N]+1)$, where $C_{N,m}$ denotes the largest number of transpositions which is necessary in changing any N-repeated-permutation of elements of $\{1,\ldots,m\}$ into the N-repeated-permutation to be arranged in ascending order of magnitude. By repeating transposition $(j_k,j_{k+1}) \to (j_{k+1},j_k)$ we get the equality with replaced $W_{\tilde{J}_\nu;\Phi_{\tilde{J}_\nu}}$ by $W_{J^0_\nu;\Phi_{J^0_\nu}}$ in (2.32). The symbol $w_{J^0_\nu}(t,t_1,\ldots,t_\nu,s)$ of $W_{J^0_\nu;\Phi_{J^0_\nu}}$ is defined by the product of $C_{N,m}([\nu/N]+1)$

elements of $\{\partial_t \omega_{j,k}\}$ and the composed function of w_{J_ν} and $C_{N,m}([\nu/N]+1)$ elements of $\{\omega_{j,k}\}$ at most. Consequently, by (2.10) We get

$$(2.34)\qquad |w_{J_\nu}(t,t_1,\dots,t_\nu,s)|_\ell^{(0)} \leqq C_{0,\ell}^{\nu} \times M^{C_{N,m}([\nu/N]+1)} \qquad (0 \leqq s \leqq t_\nu \leqq \cdots \leqq t_1 \leqq t \leqq T),$$

where $M = \max_{1 \leqq j,k \leqq m} \max_{0 \leqq s \leqq \tau \leqq t \leqq T} |\partial_\tau \omega_{j,k}(t,\tau,s)|$.

iv) From the definition of J_ν^0 and Proposition 3^0 it follows that $\Phi_{J_\nu^0}(t,t_1,\dots,t_\nu,s)$ is independent of $[(\nu+1)/2]$ elements of $\{t_j\}_{j=1}^{\nu}$ at least. Hence, for a fixed μ $(\nu/3 \leqq \mu \leqq \nu/2)$, we have

$$\Phi_{J_\nu^0}(t,t_1,\dots,t_\nu,s) = \Phi_{J_\mu}(t,t_{\nu_1},\dots,t_{\nu_\mu},s)$$

for some $J_\mu = (j_{\nu_1},\dots,j_{\nu_{\mu+1}})$. Then we get

$$I = \int_s^t \int_s^{\tilde{t}_1} \cdots \int_s^{\tilde{t}_{\mu-1}} W_{J_\mu;\Phi_{J_\mu}}(t,\tilde{t}_1,\dots,\tilde{t}_\mu,s)\, d\tilde{t}_1 \cdots d\tilde{t}_\mu ,$$

$$(\tilde{t}_j = t_{\nu_j},\ j = 1,\dots,\mu),$$

where the symbol $w_{J_\mu}(t,\tilde{t}_1,\dots,\tilde{t}_\mu,s)$ of $W_{J_\mu;\Phi_{J_\mu}}$ is determined by the integral of $w_{J_\nu^0}(t,t_1,\dots,t_\nu,s)$ with respect to $\{t_j\}_{j=1}^{\nu} \setminus \{\tilde{t}_j\}_{j=1}^{\mu}$. From the representation of integral and (2.34) we have

$$(2.35)\qquad |w_{J_\mu}(t,\tilde{t}_1,\dots,\tilde{t}_\mu,s)|_\ell^{(0)} \leqq \frac{(t-\tilde{t}_1)^{\nu_1-1}}{(\nu_1-1)!} \times \frac{(\tilde{t}_1-\tilde{t}_2)^{\nu_2-\nu_1-1}}{(\nu_2-\nu_1-1)!} \times \cdots \times \frac{(\tilde{t}_\nu-s)^{\nu-\nu_\mu}}{(\nu-\nu_\mu)!} \times |w_{J_\nu^0}|_\ell^{(0)}$$

$$\leq \frac{(t-s)^{\nu-\mu}}{(\nu-\mu)!} \times C_{0,\ell}^{\nu} \times M^{C_{N,m}(\nu/2)}$$

$$(0 \leq s \leq \tilde{t}_\mu \leq \cdots \leq \tilde{t}_1 \leq t \leq T) .$$

Here, we used $[\nu/N]+1 \leq \nu/2$. Applying the preceding procedure to the Fourier integral operator $W_{J_\mu;\Phi_{J_\mu}}$ instead of $W_{J_\nu;\Phi_{J_\nu}}$, we get the equality

$$I = \int_s^t \cdots \int_s^{t_{\gamma-1}} W_{J_\gamma;\Phi_{J_\gamma}}(t,t_1,\ldots,t_\gamma,s)dt_1\cdots dt_\gamma$$

for some $W_{J_\gamma;\Phi_{J_\gamma}}$ $(\mu/3 \leq \gamma \leq \mu/2)$ whose symbol satisfies

(2.36) $$| w_{J_\gamma}(t,t_1,\ldots,t_\gamma,s) |_\ell^{(0)}$$

$$\leq \frac{(t-s)^{\nu-\mu}}{(\nu-\mu)!} \times \frac{(t-s)^{\mu-\gamma}}{(\mu-\gamma)!} \times C_{0,\ell}^{\nu} \times M^{C_{N,m}(\nu/2 + \mu/2)}$$

Repeating this process, we finally obtain (2.28). Since $\gamma \leq \mu/2 \leq \nu(1/2)^2$, the number of transpositions to be necessary is $C_{N,m}\times\nu$ at most. This fact and (2.36) lead us to (2.29) with $\alpha = 0$ because from the Stirling formula it follows that

$$(\nu-\mu_1)!(\mu_1-\mu_2)!\cdots(\mu_n-k)! \geq C_0^{k-\nu}\times(\nu-k)!$$

for some constant $C_0 > 0$ if $3\mu_j \geq \mu_{j-1} \geq 2\mu_j$ $(j = 1,\ldots,n ,\ \mu_0 = \nu)$. Considering $\partial_t^\alpha w_{J_\nu}(t,t_1,\ldots,t_\nu,s)$ in place of $w_{J_\nu}(t,t_1,\ldots,t_\nu,s)$ and noting that the number of transpositions is constant times as many as ν , we get (2.29) with $\alpha \neq 0$. Q.E.D.

Let V be one of the Fréchet space $\mathcal{A}$, $\mathcal{B}$, $H_{-\infty}$. Then, using the fundamental solution $E(t,s)$ given in Theorem 2 and 3 we obtain

Theorem 4. i) The solution $U(t) \in \mathcal{B}_t^1(V)$ of the Cauchy problem (2.1) for $G \in V$ and $F(t) \in \mathcal{B}_t^0(V)$ exists uniquely in $[0,T]$ and can be solved by

$$U(t) = \mathbb{E}(t,0)\,G + \int_0^t \mathbb{E}(t,s)\,F(s)\,ds \quad , \tag{2.37}$$

where $\mathbb{E}(t,s) = \left\{ E_{J,J'} \; ; \; \begin{matrix} J \downarrow \\ J' \rightarrow \end{matrix} \right\}$ $(J,J' \in \Pi)$ is the fundamental solution for $\mathbb{L}$.

ii) The solution $u(t) \in \mathcal{B}_t^m(V)$ of the Cauchy problem (1.1) for $g_j \in V$ $(j = 0,\dots,m-1)$ and $f(t) \in \mathcal{B}_t^0(V)$ exists uniquely and can be solved by

$$u(t) = \sum_{J \in \Pi} E_{0,J}(t,0) \Big(g_{|J|} + \sum_{j=0}^{|J|-1} C_j^J(0) g_j \Big) + \int_s^t \Big(\sum_{J \in \Pi_{m-1}} E_{0,J}(t,s) \Big) f(s)\,ds \; . \tag{2.38}$$

Proof. i) is clear from (2.14). From (2.37) we obtain $\partial^J u \in \mathcal{B}_t^0(V)$ $(J \in \Pi)$. In view of (1.5) and (1.1) we get $u(t) \in \mathcal{B}_t^m(V)$.

Definition. We define the multiplicity of $\lambda_1,\dots,\lambda_m$ on $[0,T]\times R_x^n \times R_\xi^n$ by the smallest integer κ $(1 \leqq \kappa \leqq m)$ satisfying the following: In case $\kappa \leqq m-1$, for any $(\kappa+1)$-combination $\{j_1,\dots,j_{\kappa+1}\}$ of elements of $\{1,\dots,m\}$ and any fixed (t,x,ξ) $[0,T]\times R_x^n \times \{|\xi| \geqq M\}$ (, where M is the constant of (0.3),) there exists some pair of λ_{j_k} and $\lambda_{j_{k'}}$ $(j_k , j_{k'} \in \{j_1,\dots,j_{\kappa+1}\})$ such that

$$| \lambda_{j_k}(t,x,\xi) - \lambda_{j_{k'}}(t,x,\xi) | \geqq \delta|\xi| \tag{2.39}$$

for a positive δ . For the convinience of discussing in R_x^n

we assume that δ is independent of (t,x,ξ). The multiplicity $\kappa = m$ means that there exists no pair satisfying (2.39).

Corollary. Let κ be the multiplicity of $\lambda_1,\dots,\lambda_m$. Then the solution $u(t)$ of (1.1) belongs to $\mathcal{B}_t^0(H_{s-\kappa}) \cap \mathcal{B}_t^1(H_{s-\kappa-1}) \cap \cdots \cap \mathcal{B}_t^m(H_{s-\kappa-m})$ if $g_j \in H_{s-j-1}$ $(j = 0,\dots,m-1)$ and $f(t) \in \mathcal{B}_t^0(H_{s-m})$.

Proof. Since $E_{J,J'}$ are continuous operators from H_s into itself and $C_j^J(0) \in \mathcal{S}^{|J|-1}$, from (2.37) we obtain $\partial^J u \in \mathcal{B}_t^0(H_{s-m})$ $(J \in \Pi)$. In view of (1.5) we see that case $\kappa = m$ is clear. Other case is also obvious if we show $\partial^J u \in \mathcal{B}_t^0(H_{s-\kappa-|J|})$ $(|J| \leqq m-\kappa-1)$. Fix a point $(t_0,x^0,\xi^0) \in [0,T]\times R_x^n \times \{|\xi| \geqq M\}$ and take $\chi(t,x,D_x) \in \mathcal{B}^\infty([0,T];\mathcal{S}^0)$ such that the support of $\chi(t,x,\xi)$ is contained in a sufficiently small conic neighborhood of (t_0,x^0,ξ^0). For any $J \in \Pi_k$ $(k < m-\kappa)$ we can find $j_k, j_{k'}$ such that $(j_k, J), (j_{k'}, J) \in \Pi_{k+1}$ and the pair of λ_{j_k}, $\lambda_{j_{k'}}$ satisfies (2.39) on $\operatorname{supp}\chi \cap \{|\xi| \geqq M\}$, since (0.3) holds. Note that

$$(2.40) \qquad (\lambda_{j_k} - \lambda_{j_{k'}})\partial^J u = \partial_{j_k}\partial^J u - \partial_{j_{k'}}\partial^J u .$$

Since $(\lambda_{j_k} - \lambda_{j_{k'}})(t,x,D_x)$ is elliptic on $\operatorname{supp}\chi$ and the right side of (2.40) belongs to $\mathcal{B}_t^0(H_s)$, we see that $\chi\partial^J u \in \mathcal{B}_t^0(H_{s+1})$ $(J \in \Pi_k,\ k < m-\kappa)$. Noting that δ is independent of (t,x,ξ), by the patition of unity we obtain $\partial^J u \in \mathcal{B}_t^0(H_{s+1})$ $(J \in \Pi_k,\ k < m-\kappa)$. The fact $\partial^J u \in \mathcal{B}_t^0(H_{s+1})$ $(J \in \Pi_{m-\kappa-1})$ and (2.40) leads

us to $\partial^J u \in \mathcal{B}_t^0(H_{s+2})$ ($J \in \mathbb{I}_{m-\kappa-2}$). By repeating this process, the proof is completed. Q.E.D.

§3. Propagation of Singularities.

Let $\{X^j, \Xi^j\}(t,s;x,\xi)$ denote the solutions of

$$(3.1)\qquad \begin{cases} \dfrac{dq}{dt} = \nabla_\xi \lambda_j(t,q,p) \quad , \quad \dfrac{dp}{dt} = -\nabla_x \lambda_j(t,q,p) \\ (q,p)\big|_{t=s} = (x,\xi) \qquad (j = 1,\dots,m). \end{cases}$$

Theorem 5. Let $u(t) \in \mathcal{B}_t^m(H_{-\infty})$ be the solution of the Cauchy problem (1.1) for $g_j \in H_{-\infty}$ ($j = 0,\dots,m-1$) and $f = 0$. Assume that the multiplicity of $\lambda_1, \dots, \lambda_m$ is constant. For the brevity we assume that $\lambda_1, \dots, \lambda_r$ ($r < m$) satisfy for a constant $C_0 > 0$ and M of (0.3)

$$(3.2)\qquad |\lambda_j(t,x,\xi) - \lambda_{j'}(t,x,\xi)| \geqq C_0|\xi| \qquad (j \neq j')$$

$$\text{on } [0,T]\times R_x^n \times \{|\xi| \geqq M\}.$$

and λ_k ($k \geqq r+1$) is equal to some λ_j ($j \leqq r$). Then, we have for the wave front set $WF(u(t))$

$$(3.3)\qquad WF(u(t)) \subset \bigcup_{j=1}^{r} C_j(t) \quad ,$$

where $C_j(t) = \{ \{X^j, \Xi^j\}(t,0;y,\eta) \; ; \; (y,\eta) \in \bigcup_{k=0}^{m-1} WF(g_k) \}$.

Remark. This result was obtained by Chazarain [1]. From (0.3) we see that the right side of (3.3) defines a conic set for large $|\eta|$.

Proof. Let $\mathbb{L}$ be the hyperbolic system in (2.1). If we

exchange the column and the row, we may write $\mathbb{L}$ in the form

$$\mathbb{L} = D_t + \begin{pmatrix} \mathcal{D}_1 & & 0 \\ & \ddots & \\ 0 & & \mathcal{D}_r \end{pmatrix} + B(t) \quad , \quad \mathcal{D}_j = \begin{pmatrix} \lambda_j(t) & & \\ & \ddots & 0 \\ & 0 & \lambda_j(t) \end{pmatrix}$$

where $B(t) \in \mathcal{B}^{\infty}([0,T];\mathcal{S}^0)$. By the same way as in [6], we can find $N(t) \in \mathcal{B}^{\infty}([0,T];\mathcal{S}^0)$ such that

$$\mathbb{L}N(t) = N(t)\tilde{\mathbb{L}} \qquad (\mathrm{mod}\ \mathcal{B}^{\infty}([0,T];\mathcal{S}^{-\infty})) ,$$

where $\tilde{\mathbb{L}} = D_t + \begin{pmatrix} \tilde{\mathcal{D}}_1 & & \\ & \ddots & \\ & & \tilde{\mathcal{D}}_r \end{pmatrix}$, $\tilde{\mathcal{D}}_j(t) = \mathcal{D}_j(t) + B_j(t)$ and

$B_j(t) \in \mathcal{B}^{\infty}([0,T];\mathcal{S}^0)$. Hence, we have the fundamental solution $\mathbb{E}(t,s)$ for $\mathbb{L}$ which can be represented in the form

$$\mathbb{E}(t,s) = \sum_{j=1}^{r} W_{\phi_j}(t,s) + W_{-\infty}(t,s) \quad (W_{-\infty}(t,s) \in \mathcal{B}^{\infty}(\Delta;\mathcal{S}^{-\infty})) , \tag{3.4}$$

where $W_{\phi_j}(t,s)$ are Fourier integral operators with phase functions $\phi_j(t,s)$ and symbols $w_j(t,s) \in \mathcal{B}^{\infty}(\Delta;S^0)$ (See Theorem 2.5 and its remark of [6]). This representation leads us to (3.3).

Q.E.D.

Now, according to [8] we define the trajectory $\{Q_{J_\nu}, P_{J_\nu}\}(t,t_1,\ldots,t_\nu;y,\eta)$ $(\nu = 0,1,\ldots)$ for a fixed $J_\nu = (j_1,\ldots,j_{\nu+1}) \in \mathcal{J}_{\nu+1}$, $\{t_1,\ldots,t_\nu\}$ $(0 \leqq t_\nu \leqq \cdots \leqq t_1 \leqq t \leqq T)$ and $(y,\eta) \in R^{2n}$ as follows: First we define $\{Q_{j_{\nu+1}}, P_{j_{\nu+1}}\}(t;y,\eta)$ as the solution of (3.1) with $j = j_{\nu+1}$, $s = 0$ and $(x,\xi) = (y,\eta)$. Next, for $1 \leqq k \leqq \nu$ we define $\{Q_{J_{k,\nu}}, P_{J_{k,\nu}}\}(t,t_k,\ldots,t_\nu;y,\eta)$ as the solution

of (3.1) with $j = j_k$, $s = t_k$ and $(x,\xi) = \{Q_{J_{k+1,\nu}}, P_{J_{k+1,\nu}}\}$ $(t_k,\dots,t_\nu;y,\eta)$, where $J_{k,\nu}$ denotes $(j_k,\dots,j_{\nu+1})$.

For J_ν, (y,η) and a fixed $0 \leqq \varepsilon < 1$ we define an ε-station-chain $\{t_1,\dots,t_\nu\}$ as the points $T \geqq t > t_1 > \cdots > t_\nu > 0$ such that

$$(3.5)\quad \left| \lambda_{j_k}(t_k,x^k,\xi^k) - \lambda_{j_{k+1}}(t_k,x^k,\xi^k) \right| \leqq \varepsilon|\eta|$$

$$\text{at } (x^k,\xi^k) = \{Q_{J_{k+1,\nu}}, P_{J_{k+1,\nu}}\}(t_k,\dots,t_\nu;y,\eta)$$

$$(k = 1,\dots,\nu)$$

and define the ε-station-set $\Lambda_{\varepsilon,J_\nu}(y,\eta)$ by the set of all ε-station-chains $\{t_1,\dots,t_\nu\}$.

Furthermore, we set

$$(3.6)\quad \Lambda_\varepsilon(t,y,\eta) = \{ \{Q_{J_\nu}, P_{J_\nu}\}(t,t_1,\dots,t_\nu;y,\eta) ;\ \{t_1,\dots,t_\nu\} \in \Lambda_{\varepsilon,J_\nu}(y,\eta),\ J_\nu \in \mathbb{N}_{\nu+1},\ \nu = 0,1,\dots \}$$

Then, if we apply Theorem 3.4 of [8] to the hyperbolic system (2.1), by Theorem 4 we have directly the following

Theorem 6^0. Let $u(t) \in \mathcal{B}^m_t(H_{-\infty})$ be the solution of the Cauchy problem (1.1) for $g_j \in H_{-\infty}$ $(j = 0,\dots,m-1)$ and $f(t) = 0$. Set

$$(3.7)\quad \Gamma_{t,\varepsilon} = \{ \Lambda_\varepsilon(t,y,\eta);\ (y,\eta) \in \bigcup_{j=0}^{m-1} WF_\varepsilon(g_j) \} ,$$

where $WF_\varepsilon(g_j)$ is defined by

$$WF_\varepsilon(g_j) = \{(y,\eta) ;\ \operatorname{dis}\{(y,|\eta|^{-1}\eta) , WF(g_j)\} \leqq \varepsilon \} .$$

Then, $\Gamma_t = \bigcap_{0<\varepsilon<1} \Gamma_{t,\varepsilon}$ is closed, and we have

(3.8) $\quad WF(u(t)) \subset \Gamma_t$.

If we note the assumption (0.2) carefully, we can get a sharp form of Theorem 6^0 , which is more convinient in order to investigate $WF(u(t))$ as follows.

Theorem 6. Let $u(t) \in \mathcal{B}^m_t(H_{-\infty})$ be the solution of the Cauchy problem (1.1) for $g_j \in H_{-\infty}$ ($j = 0,\dots,m-1$) and $f(t) = 0$. Set

(3.9) $\quad \Lambda'(t,y,\eta) = \{ \{Q_{J_k}, P_{J_k}\}(t,t_1,\dots,t_k;y,\eta)$;

$\{t_1,\dots,t_k\}$ ($0 < t_k < \cdots < t_1 < t < T$),

$J_k = (j_1,\dots,j_{k+1}) \in \tilde{\Pi}_{k+1}$ such that

$\lambda_{j_1} = \lambda_{j_2} = \cdots = \lambda_{j_{k+1}}$ at $(0,y,\eta)$,

$0 \leq k \leq m-1$ },

and set

(3.10) $\quad \Gamma'_t = \{ \Lambda'(t,y,\eta)$; $(y,\eta) \in \bigcup_{j=0}^{m-1} WF(g_j) \}$.

Then, we have

(3.11) $\quad WF(u(t)) \subset \Gamma'_t$

Proof. We first show that

(3.12) $\quad WF(u(t)) \subset \tilde{\Gamma}_t = \overline{\{ \Lambda_0(t,y,\eta)$; $(y,\eta) \in \bigcup_{j=0}^{m-1} WF(g_j) \}}$,

where $\overline{\{\quad\}}$ denotes the closure of $\{\quad\}$ and $\Lambda_0(t,y,\eta)$ is defined by (3.6) with $\varepsilon = 0$.

By using the equivalent condition (2.18) to (0.2), we obtain

$$(3.13)\qquad \frac{d}{d\sigma}\{\lambda_j(\sigma,y(\sigma),\eta(\sigma)) - \lambda_k(\sigma,y(\sigma),\eta(\sigma))\}$$
$$= \tilde{a}_{j,k}(\sigma,y(\sigma),\eta(\sigma))\{\lambda_j(\sigma,y(\sigma),\eta(\sigma)) - \lambda_k(\sigma,y(\sigma),\eta(\sigma))\},$$

where $(y(\sigma),\eta(\sigma))$ is the bicharacteristics curve of λ_j. Since $\tilde{a}_{j,k}$ is bounded in $[0,T]\times R^{2n}$, it follows from (3.13) that

$$(3.14)\qquad |\lambda_j(t,X^j,\Xi^j) - \lambda_k(t,X^j,\Xi^j)|$$
$$\leqq e^{c_0(t-s)}|\lambda_j(s,x,\xi) - \lambda_k(s,x,\xi)| \quad (c_0 = \max \tilde{a}_{j,k}),$$

where $(X^j,\Xi^j) = \{X^j,\Xi^j\}(t,s;x,\xi)$. Let $J_{\nu,k}$ denote $(j_1,\dots,j_k)$ for a $J_\nu = (j_1,\dots,j_{\nu+1}) \in \mathcal{K}_{\nu+1}$. For a fixed $J_\nu \in \mathcal{K}_{\nu+1}$, (t_0,x^0,ξ^0) and $\{t_1,\dots,t_\nu\}$ we define $\{Y,\P\}(J_{\nu,k};t_0,t_1,\dots,t_{k-1},t;x^0,\xi^0)$ $(1\leqq k\leqq \nu+1)$ as follows: First $\{Y,\P\}(j_1;t_0,t_1;x^0,\xi^0)$ is defined as the solution of (3.1) with $j = j_1$ and $(x,\xi) = (x^0,\xi^0)$. Next, for $2\leqq k\leqq \nu+1$, $\{Y,\P\}(J_{\nu,k};t_0,\dots,t_{k-1},t;x^0,\xi^0)$ is defined as the solution of (3.1) with $j = j_k$ and $(x,\xi) = \{Y,\P\}(J_{\nu,k-1};t_0,\dots,t_{k-1};x^0,\xi^0)$. Furthermore we define the curve $C(J_{\nu,k};t_0,t_1,\dots,t_k;x^0,\xi^0) = \{(X(\sigma),\Xi(\sigma));$ $t_{k+1}\leqq\sigma\leqq t_0\}$ $(k = 1,\dots,\nu)$ by

$$(X(\sigma),\Xi(\sigma)) = \{Y,\P\}(J_{\nu,\ell};t_0,\dots,t_{\ell-1},\sigma;x^0,\xi^0)$$
$$\text{on } t_{\ell-1}\leqq\sigma\leqq t_\ell \quad (\ell = 1,\dots,k+1,\ t_{\nu+1} = 0).$$

If we set

$$(3.15)\qquad (y^0,\eta^0) = \{Y,\P\}(J_\nu;t_0,\dots,t_\nu,0;x^0,\xi^0),$$

then we have

$$(3.16)\qquad \{Y\,,\P\}(J_{\nu,k};t_0,\dots,t_k;x^0,\xi^0)$$

$$= \{Q_{J_{k+1,\nu}}\,,P_{J_{k+1,\nu}}\}(t_k,\dots,t_\nu;y^0,\eta^0)\quad (k=1,\dots,\nu).$$

If we assume that $(x^0,\xi^0)\notin \hat{\Gamma}_{t_0}$, then we can find some $\varepsilon_0>0$ satisfying the following: For any J_ν and $\{t_1,\dots,t_\nu\}$ we have

$$(3.17)\qquad (y^0,\eta^0)\in WF_{\varepsilon_0}(G)\ (\ni WF_{\varepsilon_0}(g_j))$$

$$\Rightarrow \quad {}^{\exists}(X^k_\nu,\Xi^k_\nu) \text{ such that}$$

$$|\lambda_{j_k}(t_k,X^k_\nu,\Xi^k_\nu)-\lambda_{j_{k+1}}(t_k,X^k_\nu,\Xi^k_\nu)|\geq \varepsilon_0|\eta^0|\,,$$

where (y^0,η^0) is defined by (3.15) and (X^k_ν,Ξ^k_ν) $(k=1,\dots,\nu)$ are defined by the right side of (3.16). Indeed, if we deny (3.17), then for any positive integer γ there exist some $J^\gamma_{\nu_\gamma}=(j_1,\dots,j_{\nu_\gamma+1})$ and some $1/\gamma$-station-chain $\{t^\gamma_1,\dots,t^\gamma_{\nu_\gamma}\}$ such that $(y^\gamma,\eta^\gamma)\in WF_{1/\gamma}(G)$, where (y^γ,η^γ) is defined by the right side of (3.15) with $\{t_1,\dots,t_\nu\}=\{t^\gamma_1,\dots,t^\gamma_{\nu_\gamma}\}$ and $J_\nu=J^\gamma_{\nu_\gamma}$. Hence we have for any γ

$$(3.18)\qquad |\lambda_{j_k}(t_k,X^k_{\nu_\gamma},\Xi^k_{\nu_\gamma})-\lambda_{j_{k+1}}(t_k,X^k_{\nu_\gamma},\Xi^k_{\nu_\gamma})|\leq \frac{1}{\gamma}|\eta^\gamma|\,,$$

where $(X^k_{\nu_\gamma},\Xi^k_{\nu_\gamma})$ are defined by the right side of (3.16) with $J_\nu=J^\gamma_{\nu_\gamma}$, $\{t_1,\dots,t_\nu\}=\{t^\gamma_1,\dots,t^\gamma_{\nu_\gamma}\}$ and $(y^0,\eta^0)=(y^\gamma,\eta^\gamma)$.

By taking a subsequence of $\{J^\gamma_{\nu_\gamma}\}_{\gamma=1}^{\infty}$ and using a new numbering

of λ_j ($j = 1,\dots,m$), if necessary, we may assume that each $J^{\gamma}_{\nu_\gamma}$ satisfies $\{j_1,\dots,j_{\nu_\gamma+1}\} = \{1,\dots,r\}$ ($r \leq m$) and is arranged as follows: $j_1 = 1$, $j_2 = 2$, $j_{k_{\gamma,3}} = 3$ and $\{j_1,\dots,j_{k_{\gamma,3}-1}\} = \{1,2\}$,..., $j_{k_{\gamma,r}} = r$ and $\{j_1,\dots,j_{k_{\gamma,r}-1}\} = \{1,\dots,r-1\}$ for some $k_{\gamma,3} < \cdots < k_{\gamma,r}$ ($\leq \nu_\gamma + 1$). Then, (3.18) with $k = 1$ and (3.14) lead us to $\lambda_1 = \lambda_2$ at (t_0,x^0,ξ^0). Since it follows from (3.14) that $\lambda_1 = \lambda_2$ on the curve $C(J^{\gamma}_{\nu_\gamma,k_{\gamma,3}-1} ; t^{\gamma}_0,\dots,t^{\gamma}_{k_{\gamma,3}-1}$ $x^0,\xi^0)$, by using (3.18) with $k = k_{\gamma,3}-1$ and (3.14) we get $\lambda_1 = \lambda_2 = \lambda_3$ at (t_0,x^0,ξ^0). Repeating this process we finally see that $\lambda_1 = \lambda_2 = \cdots = \lambda_r$ at (t_0,x^0,ξ^0) and therefore on the curve $C(J^{\gamma}_{\nu_\gamma} ; t^{\gamma}_0,\dots,t^{\gamma}_{\nu_\gamma} ; x^0,\xi^0)$. By taking a subsequence, if necessary, we may assume that $(y^{\gamma},\eta^{\gamma})$ converges to some $(y^{\infty},\eta^{\infty}) \in WF(G)$. Set $(x^{\gamma},\xi^{\gamma}) = \{Q_{J^{\gamma}_{\nu_\gamma}}, P_{J^{\gamma}_{\nu_\gamma}}\}(t,t^{\gamma}_1,\dots,t^{\gamma}_{\nu_\gamma};y^{\infty},\eta^{\infty})$. Then, it is easy to check that $(x^{\gamma},\xi^{\gamma})$ converges to (x^0,ξ^0). Since $\lambda_1 = \lambda_2 = \cdots = \lambda_r$ at $(0,y^{\infty},\eta^{\infty})$, this shows that $(x^0,\xi^0) \in \tilde{\Gamma}_{t_0}$, which is contradiction. Using (3.17) instead of (3.43) of [8], we can see that $(x^0,\xi^0) \notin WF(u(t_0))$.

If we use the following lemma, it is easy to see that $\Lambda_0(t,y,\eta) = \Lambda'(t,y,\eta)$. Hence, the proof is completed since the right side of (3.9) is closed set. Q.E.D.

Lemma 2. Assume that (2.18) holds. Given t, t_1, s ($0 \leq s$

$< t_1 < t \leqq T$) and (y,η) , if $\lambda_j = \lambda_k$ at (s,y,η) , then we can find a $\omega = \omega(t,t_1,s;y,\eta)$ ($s < \omega < t$) such that

$$(3.19) \qquad \{X_{j,k}, \Xi_{j,k}\}(t,\omega,s;y,\eta) = \{X_{k,j}, \Xi_{k,j}\}(t,t_1,s;y,\eta) .$$

Here $\{X_{j,k}, \Xi_{j,k}\}(t,t_1,t_2)$ denotes the soltion of (3.1) with $s = t_1$ and $(x,\xi) = \{X^k,\Xi^k\}(t_1,t_2;y,\eta)$.

Proof. Let H_k ($j = 1,\ldots,m$) denote the vector fields

$$\partial_t + \nabla_\xi\lambda_j\cdot\nabla_x - \nabla_x\lambda_j\cdot\nabla_\xi \qquad \text{on} \quad [0,T]\times R^n_x\times\{|\xi| \geqq M\}.$$

Set $V(t,x,\xi) = {}^t(\nabla_\xi(\lambda_j - \lambda_k) , -\nabla_x(\lambda_j - \lambda_k))$ and set $K(\sigma) = V(\sigma,q(\sigma),p(\sigma))$. Here $(q(\sigma) , p(\sigma))$ is the bicharacteristic curve of λ_j which passes through the point (y,η) when $\sigma = s$. Then from (2.18) we get

$$(3.20) \qquad \frac{d}{d\sigma} K(\sigma) = A(\sigma)K(\sigma) + (\lambda_j - \lambda_k)B(\sigma) \quad ,$$

where

$$A(\sigma) = \begin{pmatrix} \tilde{a}_{j,k} I + \nabla_\xi\nabla_x\lambda_j & \nabla_\xi\nabla_\xi\lambda_j \\ -\nabla_x\nabla_x\lambda_j & -\tilde{a}_{j,k} I + \nabla_\xi\nabla_x\lambda_j \end{pmatrix}$$

$$B(\sigma) = {}^t(\nabla_\xi\tilde{a}_{j,k} , -\nabla_x\tilde{a}_{j,k}) . \quad (I \text{ is a unit matrix. })$$

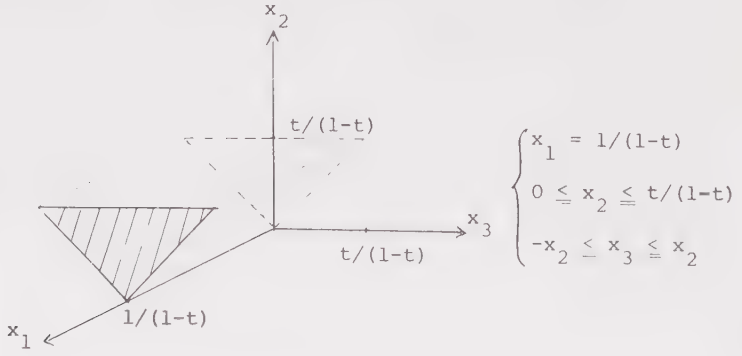
Let (s,y,η) belong to $\Sigma_{j,k} \equiv \{(t,x,\xi) \in [0,T]\times R^n_x\times\{|\xi| \geqq M\}; \lambda_j = \lambda_k\}$. Then (3.20) becomes a homogeneous differential equation for $K(\sigma)$, because $(\lambda_j - \lambda_k)(\sigma,q(\sigma),p(\sigma)) = 0$. Hence we see that $H_j = H_k$ along the flow of H_j starting at (s,y,η) if and only if $H_j = H_k$ at (s,y,η). Consequently the lemma is trivial in case $H_j = H_k$ at (s,y,η). It follows from (2.18) that

$$(3.21)\qquad [\,H_j\,,H_k\,] = \tilde{a}_{j,k}\,(\,H_k - H_j\,) + (\,\lambda_k - \lambda_j\,)(\nabla_\xi \tilde{a}_{j,k}\cdot\nabla_x - \nabla_x \tilde{a}_{j,k}\cdot\nabla_\xi\,)\,.$$

Hence, H_j and H_k form an integrable Frobenius system on $\Sigma_{j,k}$. The classical Frobenius theorem shows that the set obtained by integrating succesively H_j and H_k, starting at (s,y,η), becomes 2-dimension submanifold of $[0,T]\times R^n_x\times\{|\xi| \geq M\}$, if $H_j \neq H_k$ at (s,y,η). Therefore, the flow of H_j from $\{X_{k,j}, \Xi_{k,j}\}(t,t_1,s;y,\eta)$ and the one of H_k from (s,y,η) intersect in a point (ω,X^k,Ξ^k). Here $(X^k,\Xi^k) = \{X^k,\Xi^k\}(\omega,s;y,\eta)$. Q.E.D.

Now we explain the result of Theorem 6 by using the example:

Set $\lambda_1 = x_1^2\cdot\xi_1$, $\lambda_2 = x_1^2(\xi_1 + \xi_2 + \xi_3)$, $\lambda_3 = x_1^2(\xi_1 + \xi_2 - \xi_3)$. and assume that $\bigcup_{j=0}^{2} WF(g_j) = \{(1,0,0,\delta\eta_1^0,0,0)\ ;\ \eta_1^0 \neq 0\ ,\delta \geq 1\}$ ($x \in R^3$, $m = 3$). Then, $WF(u(t))$ ($0 < t < 1$) is contained within 3-dimension manifold $\Gamma_t' \subset R^3_x\times R^3_\xi$. The projection of Γ_t' into R^3_ξ is a half line $\{(\delta\eta_1^0(t-1)^2,0,0)\ ;\ \delta \geq 1\}$ and the projection of Γ_t' into R^3_x is indicated by the following.



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PROPAGATION OF SINGULARITIES OF SOLUTIONS FOR A HYPERBOLIC SYSTEM WITH NILPOTENT CHARACTERISTICS. I

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Introduction.

Let L be a hyperbolic $\ell\times\ell$ system of the form

$$(0.1)\qquad \begin{pmatrix} D_t & & 0 \\ & \ddots & \\ 0 & & D_t \end{pmatrix} + \begin{pmatrix} \lambda_1(t,x,D_x) & & 0 \\ & \ddots & \\ 0 & & \lambda_\ell(t,x,D_x) \end{pmatrix} + B(t,x,D_x)$$

$$\text{on } (-T,T)\times R^n_x ,$$

where $\lambda_j(t,x,\xi)$ $(j=1,\dots,\ell) \in \mathcal{B}^\infty_t(S^1)$ are real valued and satisfy $\lambda_j(t,x,c\xi) = c\lambda_j(t,x,\xi)$ for large $|\xi|$ and $c \geq 1$. Here $B(t,x,\xi) = (b_{jk}(t,x,\xi))$ $(j,k=1,\dots,\ell)$, $b_{jk}(t,x,\xi) \in \mathcal{B}^\infty_t(S^0)$. Here $a(t,x,\xi) \in \mathcal{B}^\infty_t(S^k)$ means that $a(t,x,\xi)$ belongs to a symbol class S^k of pseudo-differential operators for any fixed $t\in(-T,T)$ and, in the topology of S^k, $a(t,\cdot,\cdot)$ belongs to $\mathcal{B}^\infty$ as a

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function with respect to $t \in (-T,T)$.

We discuss the propagation of singularities of solutions for the Cauchy problem

$$\text{(C.P)} \qquad LU = 0, \quad U\Big|_{t=0} = G, \quad G \in \mathcal{E}'$$

under the nilpotent condition as follows: We say that the characteristics of L,

$$p_j(t,x,\tau,\xi) = \tau + \lambda_j(t,x,\xi), \quad j = 1,\dots,\ell,$$

satisfy the nilpotent condition (N.m) in an open set M of R^n, for a fixed integer $m \geqq 1$, if $p_1,\dots,p_\ell$ satisfy

$$\text{(N.m)} \qquad H_{j_1} H_{j_2} \cdots H_{j_m} p_{j_{m+1}}(t,x,\xi) = 0 \quad \text{in} \quad (-T,T)\times T^*(M),$$

$$(j_k = 1,2,\dots,\ell)$$

where H_j denotes the Hamilton vector field with p_j,

$$H_j = \sum_{k=0}^{n} \left(\frac{\partial p_j}{\partial \xi_k} \frac{\partial}{\partial x_k} - \frac{\partial p_j}{\partial x_k} \frac{\partial}{\partial \xi_k} \right), \quad \xi_0 = \tau, \ x_0 = t.$$

For the brevity, throughout the present paper we assume that supports of initial data G of (C.P) are contained in a fixed compact set K_0 of M such that

$$\text{(0.2)} \qquad \bigcup_{x_0 \in K_0} \{ x ; \ |x - x_0| \leqq \lambda_{\max} T \} \subset M,$$

where

$$\lambda_{\max} = \sup_{\substack{\xi \in R^n, (x,t) \in \Omega_0 \\ 1 \leqq j \leqq \ell, \ 1 \leqq k \leqq n}} |\partial_{\xi_k} \lambda_j(t,x,\xi)|, \quad \Omega_0 = M\times(-T,T).$$

First we review the case (N.1). Ludwig-Granoff[15] constructed the fundamental solution for L with $\ell = 2$. (cf. [22]). Their fundamental solution leads us to the following statement about the wave front set of the solution $U(t_0)$ of (C.P) for a fixed $t_0 \in (0,T)$;

(0.3) *If* $\delta \in T^*(M)$ *belongs to* $WF\ U(t_0)$ *then* $(t_0,\delta) \in (0,T)\times T^*(M)$ *can be connected to the set* $\{(0,\rho);\rho \in WF\ G\}$ *along the curve obtained by linking two bicharacteristic curves of* λ_1 *and* λ_2, *at most, once at the point where* λ_1 *and* λ_2 *coincide. (See Fig. 1).*

Here a curve $\{(t,x(t),\xi(t)) ; t \in (-T,T)\}$ is called a bicharacteristic curve of $\lambda(t,x,\xi)$ if $dx/dt = d_\xi\lambda(t,x,\xi)$ and $d\xi/dt = - d_x\lambda(t,x,\xi)$. For the case $\ell \geq 3$ we also have the similar statement (see [6], [12], [13]), that is,

(0.4) *If* $\delta \in WF\ U(t_0)$, *then* (t_0,δ) *can be connected to* $\{(0,\rho) ; \rho \in WF\ G\}$ *along an admissible trajectory of at most, step* $\ell - 1$ *(see Fig. 2),*

where we say that a curve obtained by linking the bicharacteristic curves of $\lambda_1,\ldots,\lambda_\ell$ k times is an admissible trajectory of step k if two characteristic roots of adjacent bicharacteristic curves have the same value at each joint. The problems of this type for L with $\ell = 2$ and $\ell \geq 3$ under the involutive condition (which is more general than the condition (N.1)) were studied by [21], [14], [2], [16] and [18].

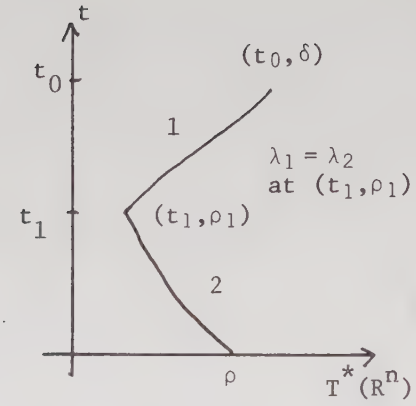


FIG. 1

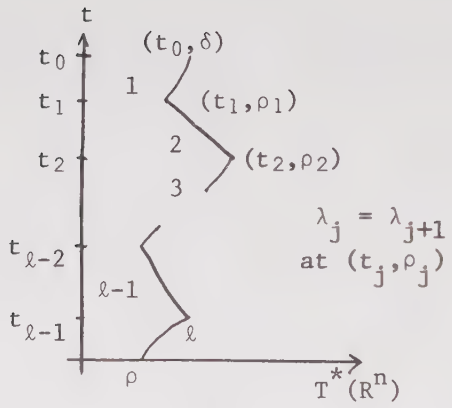


FIG. 2

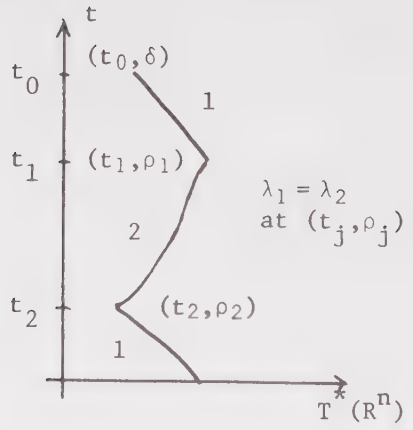


FIG. 3

Under the condition (N.2), recently Ichinose [10] showed that for the solution $U(t)$ of (C.P) with $\ell = 2$ we have the statement (0.3) with replaced *once* by *twice*, in other word,

(0.5) *WF $U(t_0)$ is contained in the set composed of the end points (at $t = t_0$) of admissible trajectories of, at most, step 2 , issuing from WF G. (See Fig.3).*

These results lead us to the following conjecture for the solution $U(t)$ of (C.P) with general ℓ under the condition (N.m):

(0.6) *There exists a constant $k = k(\ell,m)$ such that, for any $t_0 \in (0,T)$, WF $U(t_0)$ is contained in the set composed of the end points (at $t = t_0$) of admissible trajectories of, at most, step k , issuing from WF G .*

The purpose of the present paper is to consider whether (0.6) holds or not in the case (N.3) with $\ell = 2$. We shall also treat the case (N.2) with $\ell = 3$ in the forth coming paper [11]. In both cases (N.3) with $\ell = 2$ and (N.2) with $\ell = 3$, we can no longer expect (0.6) without additional conditions (*) and (**), respectively.(See §1).

In the present paper, (0.6) for the case (N.3) with $\ell = 2$ is proved under a little more restricted conditions (A_1) and (A_2) than (*). (See Theorem 1 in §1). To state the necessity of (*) we show for an example of hyperbolic systems that the wave front set of a certain solution propagates precisely along a fixed admissible trajectory. (See Theorem 2 in §1).

As in [16] and [10], we use the results of Kumano-go= Taniguchi-Tozaki [12] and Kumano-go-Taniguchi [13] where they constructed the fundamental solution for L without the condition (N.m) as the sum of Fourier integral operators with multi-phase functions, and described the propagation of singularities of solutions of (C.P). Our derivation of the statement (0.6) is carried out as the simplification of their description by means of (N.3). In such simplification, the method of Campbell-Hausdorff formula, developed in §4, plays an important role. This method gives not only another proof of Theorem of [10] (see Corollary 3.3 in §3) but also seems to be applicable to show the statement (0.6) for general case if we can find suitable additional conditions, like (A_1) (A_2) or (**).

We remark that the condition (N.m) with $m \geq 2$ is different from the involutive condition. It should be noted that our interest in the propagation of singularities is not microlocal analysis in the neighborhood of the point where $p_1 = p_2$ and $\{p_1, p_2\} \neq 0$ ($\{\cdot,\cdot\}$ denotes the Poisson bracket), but the semi-global analysis in $(-T,T)\times T^*(M)$ under the conditon (N.m) which may include both involutive and non-involutive cases.

The plan of this paper is as follows. In §1 we state our main result. In §2 we analyze the behavior of the difference of characteristic roots and investigate admissible trajectories. In §3 we prove (0.6) for the case (N.3) with $\ell = 2$ under additional conditions. §4 is devoted to prove Lemma 4.7 which plays an important role in §3 and which corresponds to Frobenius' theorem

in the involutive case. In §5 and 6 we show Theorem 2 by using a modification of the function given by Hörmander [9] as in [17]. In §7 we explain the result of the present paper by using an example.

§1. Notations and main result.

Let $\chi_j(t,s)$ ($t,s \in (-T,T)$, $j = 1,\dots,\ell$) denote the transformation;

$$(1.1) \qquad T^*(M) \ni \rho = (y,\eta) \to (x,\xi) = \chi_j(t,s)\rho \in T^*(M)$$

such that $\{(t, \chi_j(t,s)\rho) ; t \in (-T,T)\}$ is the bicharacteristic curve of λ_j through (s,ρ). For an integer $\nu \geq 0$ let $J_\nu = (j_1,j_2,\dots,j_{\nu+1})$ be a $(\nu+1)$-repeated permutation of elements $\{1,2,\dots,\ell\}$ and let Π_ν denote the set of all J_ν. Let Π_ν^0 denote the set of J_ν whose any two adjacent components are different.

Let t_0 be a fixed point in $(0,T)$ and let Δ_ν^0 denote an open subset of R^ν ($\nu \geq 1$)

$$\Delta_\nu^0 = \{ tt_\nu = (t_1,\dots,t_\nu) ; t_0 > t_1 > t_2 > \cdots > t_\nu > 0 \} .$$

A continuous curve $\{(t,x(t),\xi(t)) ; t \in [0,t_0]\}$ is called a trajectory of step ν , issuing from ρ , if for some $J_\nu \in \Pi_\nu^0$ and some $tt_\nu \in \Delta_\nu^0$ it is the bicharacteristic curve of λ_{j_q} when $t \in [t_q,t_{q-1}]$ ($q = 1,\dots,\nu+1$, $t_{\nu+1} = 0$) and $(x(0),\xi(0)) = \rho$. We often denote the trajectory by $C(J_\nu, tt_\nu ,\rho)$. A point

$$(1.2) \qquad \chi_{j_1}(t_0,t_1)\chi_{j_2}(t_1,t_2)\cdots\chi_{j_{\nu+1}}(t_\nu,0)\rho$$

is called the end point (at $t = t_0$) of the trajectory. We say that the trajectory is ε-admissible ($\varepsilon \geqq 0$) if

$$(1.3) \qquad |\lambda_{j_q}(t_q,\rho_q) - \lambda_{j_{q+1}}(t_q,\rho_q)| \leqq \varepsilon|\rho_q| \quad , \ q = 1,\dots,\nu \ ,$$

where $\rho_q = \chi_{j_{q+1}}(t_q,t_{q+1})\cdots\chi_{j_{\nu+1}}(t_\nu,0)\rho$ and $|\rho_q|$ denotes $|\xi_q|$ for $\rho_q = (x_q,\xi_q)$. We remark that a bicharacteristic curve is also called a trajectory of step 0 and it is always admissible (, 0-admissibe is called merely admissible).

For any integer $k \geqq 0$ let $\Lambda_\varepsilon^k(t_0;\rho)$ denote the set composed of the end points (at $t = t_0$) of all ε-admissible trajectories of step k, issuing from ρ. For any closed conic subset $\tilde{K}$ of $T^*(M)$, we put

$$(1.4) \qquad \Gamma^k(t_0;\tilde{K}) = \{ \Lambda_0^j(t_0;\rho) \ ; \ \rho \in \tilde{K} \ , \ j = 0,1,\dots,k \}$$

$$(1.5) \qquad \Gamma(t_0;\tilde{K}) = \bigcap_{0<\varepsilon<1} \{ \Lambda_\varepsilon^j(t_0;\rho) \ ; \ \rho \in \tilde{K}_\varepsilon \ , \ j = 0,1,\dots \} \ ,$$

where $\tilde{K}_\varepsilon$ denotes the set

$$\{ \rho \in T^*(M) \ ; \ \mathrm{dis}\,(\rho,\tilde{K}) \leqq \varepsilon \} \ .$$

First we review the result of Kumano-go-Taniguchi [13] for the Cauchy problem (C.P) without the condition (N.m).

Theorem 0. Let $U(t)$ be the solution of (C.P) with general ℓ. Then for any $t_0 \in (0,T)$ we have

$$(1.6) \qquad WF\,U(t_0) \subset \Gamma(t_0; WF\,G) \ .$$

Remark. In Theorem 3.4 of [13], (1.6) was given for t_0 in the small interval $(0,T_0]$ where the fundamental solution for L

can be represented in the sum of Fourier integral operators with multi-phase functions. However, regarding $t = T_0/2$ as the initial plane, we get (1.6) for $t_0 \in (0,3T_0/2]$ because we have such a representation of the fundamental solution in $[T_0/2,3T_0/2]$. (Note that the length T_0 of the interval can be taken uniformly in $(-T,T)$.) Repeating this process we have (1.6) for $t_0 \in (0,T)$.

Now we state our result.

Theorem 1. Assume that (N.3) holds in M. Let $U(t)$ be the solution of (C.P) with $\ell = 2$. Assume that there exists a non-positive $a(t,\rho) \in C^\infty((-T,T)\times T^*(M))$ such that

$$(A_1) \qquad H_2^2 p_1(t,\rho) = a(t,\rho) H_1^2 p_2(t,\rho) , \quad a(t,\rho) \leq 0$$

and

$$(A_2) \qquad H_j a(t,\rho) = 0 \quad (j = 1,2) \quad in \quad (-T,T)\times T^*(M).$$

Then we have for any $t_0 \in (0,T)$

$$(1.7) \qquad WF\, U(t_0) \subset \Gamma^3(t_0; WF\, G)$$

if supp $G \subset K_0$ for the compact K_0 in (0.2).

Remark 1. Though the assumption (A_2) seems to be restrictive, it is fulfilled, for instance, if we have

$$(A_3) \qquad \begin{cases} \text{The set } \{ \rho \in T^*(M) ; H_1^2 p_2(0,\rho) = 0 \} \text{ does not have} \\ \text{the inner point in } T^*(M). \end{cases}$$

Indeed, differentiating (A_1) by H_j, in view of (N.3) we have

$$H_1^2 p_2(t,\rho)H_j a(t,\rho) = 0 \quad \text{in} \quad (-T,T)\times T^*(M).$$

Since it follows from Lemma 2.1 in §2 and (A_3) that $H_1^2 p_2 \not\equiv 0$ for almost all $(t,\rho) \in (-T,T)\times T^*(M)$, we get (A_2).

Remark 2. If we define $\Gamma^k(t_0; K)$ for $t_0 \in (-T,0)$ similarily, we get (1.7) for $t_0 \in (-T,0)$.

Example 1. $\lambda_1 = 0$, $\lambda_2 = t^2\xi_1 + t\xi_2 + \xi_3$.

2. $\lambda_1 = 0$, $\lambda_2 = (t^2 - bx_1 t)\xi_2 + \xi_1$.

3. $\lambda_1 = \xi_1 + t\xi_2$, $\lambda_2 = \xi_1 + \xi_3 + (t^2 - bx_3 t)\xi_4$.

If constant $b \leq 2$, then Examples 2 and 3 satisfy (A_1) and (A_2) with $a = (b - 2)/2$. Of course, λ_j of Examples 2 and 3 are modified in the complement of $M = \{|x| < R_0\}$ for a fixed $R_0 > 0$ in order to belong $B_t^\infty(S^1)$.

The condition derived from (A_1),

$$(*) \qquad H_1^2 p_2(0,\rho)\, H_2^2 p_1(0,\rho) \leq 0 \quad , \quad \rho \in T^*(M) \ ,$$

is necessary in general for (1.7) and moreover in order to get

$$(1.8) \qquad \text{WF } U(t_0) \subset \Gamma^k(t_0 \ ; \text{ WF } G)$$

for a finite integer k . In fact, we have $H_1^2 p_2 H_2^2 p_1 = 2(b-2)\xi_2^2 > 0$ ($\xi_2 \neq 0$) for Example 2 with $b > 2$ and we have

Theorem 2. Let L_0 be a hyperbolic 2×2 system of the form (0.1) whose λ_1 and λ_2 are equal to Example 2 with $b = 4$

and $B = \sqrt{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *For any* $t_0 \in (0,T)$ *and any integer* $\mu \geqq 4$ *there exist a* $\rho_0 = (y_0, \eta_0) \in T^*(M)$ *and an admissible trajectory of step* μ, *issuing from* ρ_0, *whose end point* $\delta_0 = (x_0, \xi_0)$ *(at* $t = t_0$ *) can never be attained along other admissible trajectories, issuing from* ρ_0. *Furthermore, there exists an initial value* G *of (C.P) such that*

(1.9) $$WF\ G = \{(y_0, c\eta_0)\ ;\ c > 0\ \}$$

and

(1.10) $$WF\ U(t_0) \ni \delta_0\ ,$$

where $U(t)$ *is the solution of (C.P) with* $L = L_0$ *for* G.

In the rest of this section we announce the result for the case (N.2) with $\ell = 3$, which will be proved in the forth coming paper [11].

Theorem 3. Assume that (N.2) holds in M. *Let* $U(t)$ *be the solution of (C.P) with* $\ell = 3$. *Assume that*

(**) *for any* $\rho \in T^*(M)$, $H_1p_2(0,\rho)$, $H_2p_3(0,\rho)$ *and* $H_3p_1(0,\rho)$ *do not have the same sign simultaneously.*

Then we have for any $t_0 \in (0,T)$

(1.11) $$WF\ U(t_0) \subset \Gamma^5(t_0\ ;\ WF\ G\ )$$

if supp $G \subset K_0$ *for the compact* K_0 *in (0.2).*

Example 4. $\lambda_1 = 0$, $\lambda_2 = -t\xi_1 - \xi_2$, $\lambda_3 = (t + bx_2)\xi_1$.

This example satisfies (**) if and only if constant $b \leq 2$.
The condition (**) is necessary for (1.11) in general. Actually, for Example 4 with $b > 2$ we can obtain the statement similar to Theorem 2.

§2. Behavior of difference of characteristic roots.

Lemma 2.1. Assume that (N.3) holds in M. Then $H_1^2 p_2$ *and* $H_2^2 p_1$ *are constants on a trajectory* $C(J_\nu, tt_\nu, \rho)$ *if* $\rho \in K_0 \times R^n_\xi \setminus 0$ *for* K_0 *in (0.2).*

Proof. Put $f(t,x,\xi) = H_1^2 p_2(t,x,\xi)$. Since $H_j f = 0$ ($j = 1,2$) in $(-T,T)\times T^*(M)$ by (N.3), we have $f(t,\chi_j(t,s)\delta) = f(s,\delta)$ ($j = 1,2$) for any $\delta \in T^*(M)$, if $\chi_j(t,s)\delta \in T^*(M)$ for $s, t \in [0,T)$. Hence, for any $t \in [t_q, t_{q-1})$ $(q = 1,\dots,\nu+1,\ t_{\nu+1} = 0)$

$$f(t,\chi_{j_q}(t,t_q)\chi_{j_{q+1}}(t_q,t_{q+1})\cdots\chi_{j_{\nu+1}}(t_\nu,0)\rho)$$
$$= f(t_q,\chi_{j_{q+1}}(t_q,t_{q+1})\cdots\chi_{j_{\nu+1}}(t_\nu,0)\rho)$$
$$= \cdots = f(0,\rho) .$$

Lemma 2.2. Let $F(t;J_\nu, tt_\nu, \rho)$ *be a function defined by*

$$F(t) = (\lambda_1 - \lambda_2)(t,x(t),\xi(t)),\quad t \in [0,t_0] ,$$

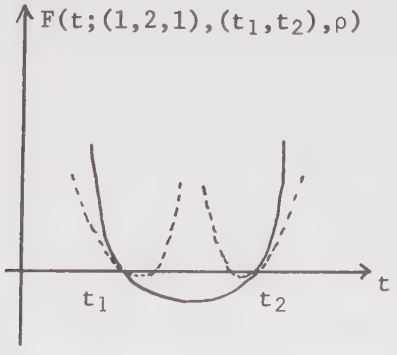
where $\{(t,x(t),\xi(t)) ;\ t \in [0,t_0]\}$ *is the trajectory* $C(J_\nu, tt_\nu, \rho)$. *Then* $F(t)$ *belongs to* C^1 *function. If (N.3) holds in M and* $\rho \in K_0 \times R^n \setminus 0$, *then* $F(t)$ *is the quadratic function on* (t_q, t_{q-1}) *whose second derivative eqauls* $-H_1^2 p_2(0,\rho)$ *or* $H_2^2 p_1(0,\rho)$ *according to* $j_q = 1$ *or* 2 .

Proof. Since d^jF/dt^j ($j = 1,2,\dots$) equals $H_1^j(p_1 - p_2)$ $= - H_1^j p_2$ or $H_2^j(p_1 - p_2) = H_2^j p_1$ on (t_q, t_{q-1}) according to $j_q = 1$ or 2 , the first assertion follows from $H_1p_2 = - H_2p_1$ and the second assertion follows from Lemma 2.1.

Lemma 2.3. Assume that (N.3) holds in M and $\rho \in T^*(M)$ belongs to $K_0 \times R^n_\xi \setminus 0$ for K_0 in (0.2). Let $(s,\delta) \in [0,t_0] \times T^*(M)$ be a point on the trajectory $C(J_\nu, tt_\nu, \rho)$.

1. If either $H_1^2 p_2(s,\delta)$ or $H_2^2 p_1(s,\delta)$ is not zero and $H_1^2 p_2(s,\delta)\, H_2^2 p_1(s,\delta) \leqq 0$, then $F(t; J_\nu, tt_\nu, \rho)$ is a convex or concave function. Hence, if the trajectory $C(J_\nu, tt_\nu, \rho)$ is admissible then its step is not more than 2 .

2. If $H_1^2 p_2(s,\delta) = H_2^2 p_1(s,\delta) = 0$, then $F(t)$ is linear in t . Furthermore, if $H_1p_2(s,\delta) = 0$ then $F(t)$ is constant. Hence, if the trajectory is admissible then its step is not more than 1 , unless $H_1p_2(s,\delta) = (p_1 - p_2)(s,\delta) = 0$.



$-H_1^2 p_2(0,\rho) > H_2^2 p_1(0,\rho) > 0$

FIG. 4

Proof. The former assertion of 1 and 2 are obvious from the preceding lemmas. We get the latters by checking the zero points of function F . (See Fig. 4).

§3. Proof of Theorem 1.

By Theorem 0, to prove Theorem 1 it suffices to show that

$$\Gamma(t_0 ; \tilde{K}) \subset \Gamma^3(t_0 ; \tilde{K}) \tag{3.1}$$

for any closed conic set $\tilde{K} \subset K_0 \times R^n_\xi \setminus 0$, where K_0 satisfies (0.2).

Suppose that $\delta \in \Gamma(t_0 ; \tilde{K})$. For any integer $\gamma = 1,2,\ldots,$ we can find a sequence of $1/\gamma$ - admissible trajectories $C_\gamma \equiv C(J_{\nu_\gamma}, tt_{\nu_\gamma}, \rho_\gamma)$ such that $\rho_\gamma \in \tilde{K}_{1/\gamma}$ and whose end points are δ. If for any γ_0 there exists a $\gamma \geqq \gamma_0$ such that $\nu_\gamma \leqq 1$, then it is easy to check that a subsequence of C_γ tends to an admissible trajectory of, at most, step 1 , issuing from some $\rho_\infty \in \tilde{K}$, and so δ belongs to $\Gamma^1(t_0 ; \tilde{K})$. We may assume that $\nu_\gamma \geqq 2$. Put $F_\gamma = F(t; J_{\nu_\gamma}, tt_{\nu_\gamma}, \rho_\gamma)$. Noting Lemma 2.3 and assumption (A_1), we consider the following cases.

Case 1. $H_1^2 p_2(t_0,\delta) H_2^2 p_1(t_0,\delta) < 0$. By changing F_γ to $-F_\gamma$, if necessary, we may assume that F_γ is a strictly convex function, that is, $d^2F_\gamma/dt^2 > 0$. Let $s_\gamma \in [0,t_0]$ be a minimum point of F_γ . If two components t_q^γ , $t_{q'}^\gamma$ of tt_{ν_γ} belong to $[0,s_\gamma]$ or $(s_\gamma,t_0]$ then we get

$$|t_q^\gamma - t_{q'}^\gamma| \leq 2(c_0\gamma)^{-1/2} \tag{3.2}$$

where $c_0 = \min (-H_1^2 p_2(t_0,\delta), H_2^2 p_1(t_0,\delta))$. Indeed, this

follows from the fact that

$$|F_\gamma(t)| \leqq 1/\gamma \quad \text{if} \quad t = t_q^\gamma \,,\ t_q^\gamma,$$

and for $t,\ s \in [0,s_\gamma]$ or $(s_\gamma,t_0]$

$$|F_\gamma(t) - F_\gamma(s)| \geqq c_0(t-s)^2/2 \,. \tag{3.3}$$

By taking a subsequence of C_γ, if necessary, we may assume that, when γ tends to ∞, t_1^γ and $t_{\nu_\gamma}^\gamma$ converge t_1^∞ and t_2^∞ ($0 \leqq t_2^\infty \leqq t_1^\infty \leqq t_0$), respectively. It follows from (3.2) that any t_q^γ ($q = 1,\dots,\nu_\gamma$) converges t_1^∞ or t_2^∞. Hence, we see that C_γ tends to some admissible trajectory of, at most, step 2, issuing from some $\rho_\infty \in \tilde{K}$. Therefore $\delta \in \Gamma^2(t_0\,;\,\tilde{K})$.

Case 2. $H_1^2 p_2(t_0,\delta) \neq 0$ but $H_2^2 p_1(t_0,\delta) = 0$. As in Case 1 we may assume that $d^2F_\gamma/dt^2 \geqq 0$. Let F_γ take a minimum value at $s_\gamma \in [0,t_0]$. If the minimum value of F_γ is smaller than $-2/\sqrt{\gamma}$ for large γ, then we can apply the preceding argument. In fact, putting $I_\gamma = \{t \in [0,t_0]\ ;\ F_\gamma(t) \geqq -1/\gamma\}$ we have

$$|dF_\gamma/dt| \geqq (1/t_0)\gamma^{-1/2} \quad \text{on} \quad I_\gamma \quad \text{(see Fig. 5)},$$

and we have

$$|F_\gamma(t) - F_\gamma(s)| \geqq (1/t_0)\gamma^{-1/2}|t-s|$$

if $t,\ s \in [0,s_\gamma] \cap I_\gamma$ or $(s_\gamma,t_0] \cap I_\gamma$. Using this instead of (3.3) we get (3.2) with c_0 replaced by $(1/t_0)^2$ and consequently we see $\delta \in \Gamma^2(t_0\,;\,\tilde{K})$. Hence, by taking a subsequence of C_γ , we may assume that for any γ

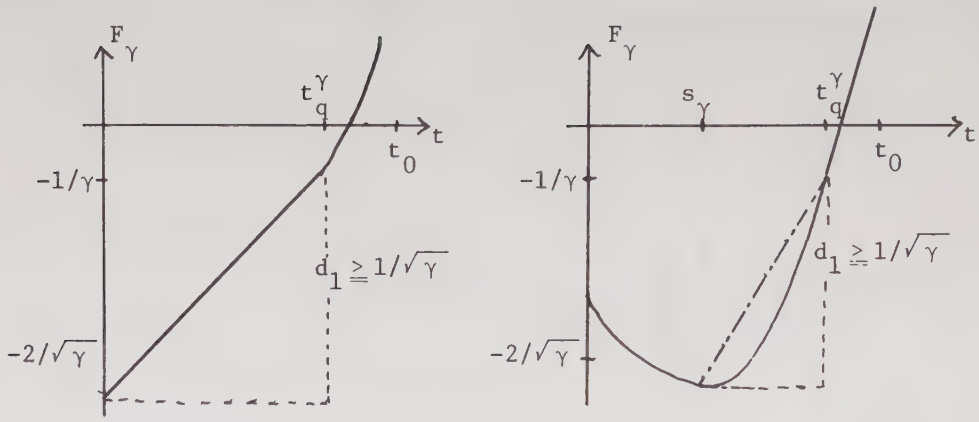


FIG. 5

(3.4) $\min F_\gamma \geqq -2/\sqrt{\gamma}$.

We also assume that, for all γ, ν_γ are even or odd, and $J_{\nu_\gamma} = (2,1,2,\ldots)$ or $(1,2,1,\ldots)$. For instance, suppose that ν_γ are even and $J_{\nu_\gamma} = (2,1,\ldots,2)$. Then we have

$$\text{(3.5)} \qquad \sum_{k=1}^{\nu_\gamma/2} |t^\gamma_{2k-1} - t^\gamma_{2k}| \leqq 4(c_1\gamma)^{-1/2} + 2\sqrt{6}(c_1\sqrt{\gamma})^{-1/2} ,$$

where $c_1 = -H_1^2 p_2(t_0,\delta) > 0$. In fact, if $t^\gamma_{2k_\gamma+1} \leqq s_\gamma < t^\gamma_{2k_\gamma}$ for some k_γ we resolve the left hand side to

$$\sum_{k<k_\gamma} + |t^\gamma_{2k_\gamma-1} - t^\gamma_{2k_\gamma}| + \sum_{k>k_\gamma} \equiv T_1 + T_2 + T_3 .$$

It follows from (3.4) that

$$T_2 = |t^\gamma_{2k_\gamma-1} - t^\gamma_{2k_\gamma}| \leqq 2\sqrt{6}(c_1\sqrt{\gamma})^{-1/2} \quad \text{(See Fig. 6).}$$

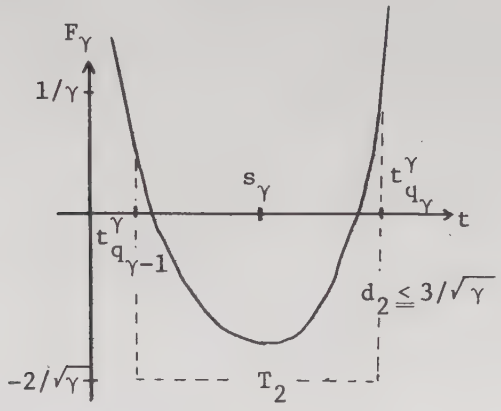


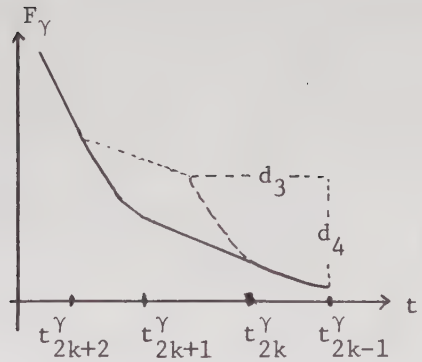
FIG. 6

On the other hand, we have

$$c_1 T_1^2/2 \leqq F_\gamma(t^\gamma_{\nu_\gamma}) - F_\gamma(t^\gamma_{2k_\gamma}) \leqq 2/\gamma$$

and also $c_1 T_3^2/2 \leqq 2/\gamma$.(See Fig. 7). Unless $t^\gamma_{2k_\gamma+1} \leqq s_\gamma < t^\gamma_{2k_\gamma}$, by adding T_2 to T_1 or T_3 , we can show (3.5), similarly. As in Case 1 let t^γ_1 and $t^\gamma_{\nu_\gamma}$ converge t^∞_1 and t^∞_2 ($0 \leqq t^\infty_2 \leqq t^\infty_1 \leqq t_0$), respectively. In view of (3.5) we can see that C_γ tends to an admissible trajectory $C_\infty((2,1,2),(t^\infty_1,t^\infty_2),\rho_\infty)$ for some $\rho_\infty \in \tilde{K}$ (with convention that the step of C_∞ is smaller than 2 unless $0 < t^\infty_2 < t^\infty_1 < t_0$). Hence, $\delta \in \Gamma^2(t_0;\tilde{K})$. For other three cases, $J_{\nu_\gamma} = (2,1,\ldots,1)$, $(1,2,\ldots,2)$ and $(1,2,1,\ldots,1)$, we can get the inequality corresponding to (3.5) and see $\delta \in \Gamma^2(t_0;\tilde{K})$.

Case 3. $H_1^2 p_2(t_0,\delta) = H_2^2 p_1(t_0,\delta) = 0$. Putting $c_2 = H_1 p_2(t_0,\delta)$ and $c_3 = (p_1 - p_2)(t_0,\delta)$, we get



$d_3 = t^\gamma_{2k-1} - t^\gamma_{2k} + t^\gamma_{2k+1} - t^\gamma_{2k+2}$

$d_4 \leqq 2/\gamma$.

FIG. 7

$$F_\gamma(t) = c_2(t - t_0) + c_3, \quad t \in [0,t_0] .$$

Since $|F_\gamma| \leqq 1/\gamma$ on $[t^\gamma_1, t^\gamma_{\nu_\gamma}]$, we get

$$|t^\gamma_1 - t^\gamma_{\nu_\gamma}| \leqq (c_2\gamma)^{-1}$$

if $c_2 \neq 0$. Hence, by taking a subsequence of C_γ , if necessary, we see that if $c_2 \neq 0$ then any t^γ_q ($q = 1,\ldots,\nu_\gamma$) converges a $t^\infty \in [0,t_0]$. Therefore, if $c_2 \neq 0$ then C_γ tends to some admissible trajectory of, at most, step 1, issuing from some $\rho_\infty \in \tilde{K}$, and so $\delta \in \Gamma^1(t_0; \tilde{K})$. If $c_2 = 0$ but $c_3 \neq 0$ then $\nu_\gamma = 0$ for large γ. Obviously $\delta \in \Gamma^0(t_0 ; \tilde{K})$.

By the above discussion we may only consider the case for $H_1^2 p_2 = H_2^2 p_1 = H_1 p_2 = H_2 p_1 = p_1 - p_2 = 0$ at (t_0,δ). In this case, all trajectory through (t_0,δ) are admissible. The proof is completed by Lemma 3.1 below. Actually, in view of Lemma 2.1,

it follows from Lemma 3.1 that for any trajectory C_γ there exists a trajectory $\tilde{C}_\gamma$ of, at most, step 3, linking (t_0,δ) and $(0,\rho_\gamma)$. For instance, we see by (3.6) below that

$$\begin{aligned}\delta &= \chi_1(t_0,t_1^\gamma)\chi_2(t_1^\gamma,t_2^\gamma)\chi_1(t_2^\gamma,t_3^\gamma)\chi_2(t_3^\gamma,t_4^\gamma)\chi_1(t_4^\gamma,0)\rho_\gamma \\ &= \chi_2(t_0,\tilde{t}_1^\gamma)\chi_1(\tilde{t}_1^\gamma,\tilde{t}_2^\gamma)\chi_2(\tilde{t}_2^\gamma,\tilde{t}_3^\gamma)\chi_1(\tilde{t}_3^\gamma,t_4^\gamma)\chi_1(t_4^\gamma,0)\rho_\gamma \\ &= \chi_2(t_0,\tilde{t}_1^\gamma)\chi_1(\tilde{t}_1^\gamma,\tilde{t}_2^\gamma)\chi_2(\tilde{t}_2^\gamma,\tilde{t}_3^\gamma)\chi_1(\tilde{t}_3^\gamma,0)\rho_\gamma \quad .\end{aligned}$$

By taking a subsequence of $\tilde{C}_\gamma$, we see that when γ tends to ∞ $\tilde{C}_\gamma$ tends to an admissible trajectory of, at most, step 3, issuing from some $\rho_\infty \in \tilde{K}$. Therefore, $\delta \in \Gamma^3(t_0;\tilde{K})$.

Lemma 3.1. *Assume that (N.3) holds in M and $\rho \in K_0 \times R^n_\xi \setminus 0$ for K_0 in (0.2). Assume (A_1) and (A_2). Then for any t_j ($j = 0,\dots,4$) with $0 \leqq t_4 < t_3 < t_2 < t_1 < t_0 < T$ there exist some $\tilde{t}_j$ ($j = 1,2,3$) with $t_4 < \tilde{t}_3 < \tilde{t}_2 < \tilde{t}_1 < t_0$ such that*

$$\begin{aligned}(3.6)\qquad &\chi_1(t_0,t_1)\chi_2(t_1,t_2)\chi_1(t_2,t_3)\chi_2(t_3,t_4)\rho \\ &= \chi_2(t_0,\tilde{t}_1)\chi_1(\tilde{t}_1,\tilde{t}_2)\chi_2(\tilde{t}_2,\tilde{t}_3)\chi_1(\tilde{t}_3,t_4)\rho\end{aligned}$$

if $H_1^2 p_2(t_4,\rho) = 0$. We also have (3.6) with χ_1 and χ_2 exchanged. (See Fig. 8).

Proof. This lemma follows from Lemma 4.7 in the next section if we put $\Omega = T^*((-T,T)\times M)$, $X = H_1$ and $Y = H_2$. Actually, noting that for $f, g \in C^2(T^*((-T,T)\times M))$

$$H_{\{f,g\}} = H_f H_g - H_g H_f \qquad (\{f,g\} = H_f g),$$

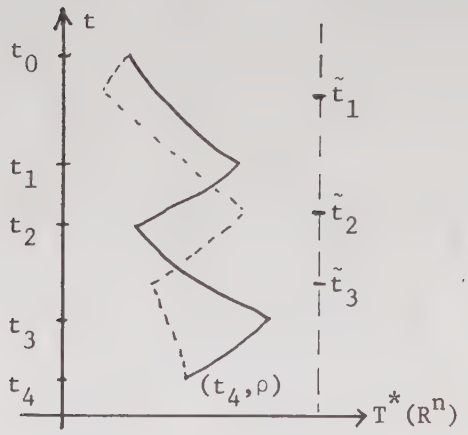


FIG. 8

$$H_{fg} = fH_g + gH_f ,$$

we see from (N.3), (A_1) and (A_2) that the assumptions of Lemma 4.7 are fulfilled with $W = (H_1^2 p_2)H_a$. Hence (3.6) follows from (4.21) because for $t, s \in (-T,T)$ and $\rho \in T^*(M)$

$$(t,\chi_1(t,s)\rho) = \pi_0 X^{t-s} \pi_0^{-1} (s,\rho)$$

$$(t,\chi_2(t,s)\rho) = \pi_0 Y^{t-s} \pi_0^{-1} (s,\rho) ,$$

where π_0 denotes the natural projection from $T^*((-T,T)\times M)$ to $(-T,T)\times T^*(M)$. The last assertion follows from that of Lemma 4.7.

Remark. By means of (3.6) we can weeken (1.7) to

(1.7)' $$WF\, U(t_0) \subset \tilde{\Gamma}^3(t_0; WF\, G) ,$$

where $\tilde{\Gamma}^3$ is defined by the right hand side of (1.4) with $k = 3$ with replaced $\Lambda_0^3(t_0,\rho)$ by the set composed of the end points of admissible trajectories corresponding to only $J_3 = (2,1,2,1)$.

In view of Lemma 2.1 we obviously obtain the following corollary from the arguments in Cases 1 and 2.

Corollary 3.2. *Let $U(t)$ be the solution of (C.P) with $\ell = 2$ under the condition (N.3). Put*

$$\omega_1 = \{ \rho \in T^*(M) \ ; \ H_1^2 p_2(0,\rho) H_2^2 p_1(0,\rho) \leqq 0 \}$$

$$\omega_2 = \{ \rho \in T^*(M) \ ; \ H_1^2 p_2(0,\rho) = H_2^2 p_1(0,\rho) = 0 \} .$$

Assume that the initial value G of (C.P) satisfies

$$WF\ G \subset (\omega_1 \setminus \omega_2)^0 \cap K_0 \times R^n_\xi \setminus 0 \tag{3.7}$$

Here ω^0 denotes the interior of ω. Then we have any $t_0 \in (0,T)$

$$WF\ U(t_0) \subset \Gamma^2(t_0 \ ; \ WF\ G). \tag{3.8}$$

From the argument of Case 3 and Lemma 4.8 in the next section we get an extension of Theorem of Ichinose [10].

Corollary 3.3. *Let $U(t)$ be the solution of (C.P) with $\ell = 2$ under the condition (N.3). Assume that the initial value G of (C.P) satisfies*

$$WF\ G \subset \omega_2^0 \cap K_0 \times R^n_\xi \setminus 0 . \tag{3.9}$$

Then we have for any $t_0 \in (0,T)$

$$WF\ U(t_0) \subset \tilde{\Gamma}^2(t_0; \ WF\ G), \tag{3.10}$$

where $\tilde{\Gamma}^2$ is defined by the right hand side of (1.4) with $k = 2$ with replaced $\Lambda_0^2(t_0,\rho)$ by the set composed of the end points of admissible trajectories corresponding to only $J_2 = (2,1,2)$.

Proof. Suppose that $\delta \in \Gamma(t_0; \tilde{K})$ ($\tilde{K} = WF\ G$). Since it follows from (3.9) that $\rho_\gamma \in \omega_2^0$ for large γ , by Lemma 2.1 we may only consider Case 3. It follows from the argument in Case 3 that we get $\delta \in \Gamma^1(t_0 ; \tilde{K})$ unless $c_2 = c_3 = 0$. Put

$$\omega = \pi_0^{-1} \{ (t,x,\xi) \in (-T,T)\times T^*(M);\ H_1^2 p_2 = H_2^2 p_1 = 0 \} .$$

If we take ω^0 ($\subset T^*((-T,T)\times M)$) and $\pi_0^{-1}(t,\Gamma(t ; \tilde{K}))$ ($t \in [0,T)$) as Ω and K of Lemma 4.8, respectively, then the hypotheses of Lemma 4.8 are satisfied. In fact, since it follows from (N.3) that

$$V(\sigma)(H_1^2 p_2) = V(\sigma)(H_2^2 p_1) = \tilde{V}(\sigma)(H_1^2 p_2) = \tilde{V}(\sigma)(H_2^2 p_1) = 0 \quad \text{in } (-T,T)\times T^*(M),$$

transformations $V(\sigma)^\tau$ and $\tilde{V}(\sigma)^\tau$ from K to Ω are well-defined for $|\tau| < \tau_0$ ($\tau_0 > 1$). Other hypotheses are checked as in the proof of Lemma 3.1. It follows from (4.33) that for any t_j ($j = 1,2,3$) with $0 \leq t_3 < t_2 < t_1 < t_0$, we obtain

$$\chi_1(t_0,t_1)\chi_2(t_1,t_2)\chi_1(t_2,t_3)\rho = \chi_2(t_0,\tilde{t}_1)\chi_1(\tilde{t}_1,\tilde{t}_2)\chi_2(\tilde{t}_2,t_3)\rho \ , \ \rho \in \Gamma(t_3 ; \tilde{K}) . \tag{3.11}$$

where $\tilde{t}_1 = t_0 - (t_1 - t_2)(t_2 - t_3)/(t_0 - t_1 + t_2 - t_3)$ and $\tilde{t}_2 = t_3 + (t_0 - t_1)(t_1 - t_2)/(t_0 - t_1 + t_2 - t_3)$. Using (3.11) we see that $\delta \in \tilde{\Gamma}^2(t_0 ; \tilde{K})$ when $c_2 = c_3 = 0$.

§4. Method of Campbell-Hausdorff formula.

Let Ω be an open set of R_x^n , K a compact subset of Ω , and let $X \in T(\Omega)$. Let f be the solution of the initial value

problem

$$\frac{df(t,x)}{dt} = X(f(t,x)) , \quad f(0,x) = x . \tag{4.1}$$

It is clear that f is a C^∞ function from $K\times(-t_0,t_0)$ to Ω for some positive number t_0 depending on K and X. Then, we say that the transformation X^t from K to Ω

$$X^t ; \quad x \to f(t,x) \tag{4.2}$$

is well-defined for $|t| < t_0$. When $|t|+|s| < t_0$ we have the group property

$$X^tX^s x = X^{t+s} x , \quad x \in K .$$

Let $Y \in T(\Omega)$ and let $g(t,x)$ be the solution of (4.1) with X replaced by Y. It is also clear that $f(t,g(s,x))$ is a C^∞ function from $K\times\{(t,s) ; |t|+|s| < t_0' \}$ to Ω for some positive number t_0' depending on K, X and Y. Then we say that the product transformation X^tY^s from K to Ω

$$X^tY^s ; \quad x \to f(t,g(s,x))$$

is well-defined for $|t|+|s| < t_0'$. Since a transformation Y^sX^t is not equal to X^tY^s in general, the transformation Y^sX^t from K to Ω is not well-defined for $|t|+|s| < t_0'$ in general. If u is a function in Ω we set

$$(e^{tX}u)(x) = u(X^tx) . \tag{4.3}$$

When $|t| < t_0$ this defines a mapping from $C_0^\infty(K)$ to $C_0^\infty(\Omega)$. If $u \in C_0^\infty(K)$ we have for $|t|+|s| < t_0$,

$$e^{tX} e^{sX} u = e^{(t+s)X} u$$

and moreover we have

$$\frac{d(\, e^{tX} u\,)}{dt} = e^{tX} X u = X e^{tX} u \ , \tag{4.4}$$

(see p. 156 of [8]).

Let $(\mathrm{ad}\ X)Y$ denote $[X,Y] = XY - YX$ for $X, Y \in T(\Omega)$ and let $(\mathrm{ad}\ X)^j Y = (\mathrm{ad}\ X)((\mathrm{ad}\ X)^{j-1} Y)$ for a positive integer j , where $(\mathrm{ad}\ X)^0 Y = Y$.

Lemma 4.1. *Let X and Y belong to $T(\Omega)$ and satisfy for some positive integer ℓ*

$$(\mathrm{ad}\ X)^{\ell} Y = 0 \quad \text{in } T(\Omega) \ . \tag{4.5}$$

For a compact K of Ω let the transformation X^t from K to Ω be well-defined for $|t| < t_0$. Then we have for $|t| < t_0$

$$e^{tX} Y e^{-tX} u = \sum_{j=0}^{\ell-1} \frac{t^j}{j!} (\mathrm{ad}\ X)^j Y u \ , \quad u \in C_0^{\infty}(K) \ . \tag{4.6}$$

Proof. By means of (4.4) we have

$$\frac{d}{dt}(\, e^{tX} Y e^{-tX} u\,) = e^{tX} (\mathrm{ad}\ X) Y e^{-tX} u \ ,$$

and moreover for any positive integer j

$$\frac{d^j}{dt^j}(\, e^{tX} Y e^{-tX} u\,) = e^{tX} (\mathrm{ad}\ X)^j Y e^{-tX} u$$

In view of (4.5), $e^{tX} Y e^{-tX} u$ is the polinomial of degree $\ell-1$ with respect to t , and its derivative of order j equals $(\mathrm{ad}\ X)^j Y$ at $t = 0$.

Lemma 4.2. *Let* $X, Y \in T(\Omega)$ *satisfy*

$$[A_{i_1}, [A_{i_2}, [A_{i_3}, A_{i_4}]]] = 0 \ , \ A_{i_j} = X, Y \ . \tag{4.7}$$

For a compact K *of* Ω *let the product transformation* $X^t Y^s$ *from* K *to* Ω *be well-defined for* $|t|+|s| < t_0$. *If* $Z(t,s)$ *denotes* $tX + sY + \frac{ts}{2}[X,Y] + \frac{1}{12}(t^2 s[X,[X,Y]] + ts^2[Y,[Y,X]])$, *then we have for* $|t|+|s| < t_0$

$$e^{tX} e^{sY} u = e^{Z(t,s)} u \ , \quad u \in C_0^\infty(K) \ , \tag{4.8}$$

provided that the trasformation $Z(t,s)^\tau$ *from* K *to* Ω *is well-defined for* $|\tau| < \tau_0$ ($\tau_0 > 1$). (cf. Campbell-Hausdorff formula in [4], [7].)

Proof. Putting $h(t,s,x) = (e^{tX} e^{sY} u)(x)$, h is a C^∞ function in $\{|t|+|s| < t_0\}\times\Omega$ and satisfy

$$\frac{dh}{dt} = Xh \ , \quad h\Big|_{t=0} = e^{sY} u \ . \tag{4.9}$$

Putting $g(\tau,t,s,x) = (e^{\tau Z(t,s)} u)(x)$, g is a C^∞ function in $\{|\tau| < \tau_0\}\times\{|t|+|s| < t_0\}\times\Omega$. We shall show that $g(1,t,s,x)$ also satisfies (4.9). From the definition of g it follows that

$$\frac{dg}{d\tau} = Z(t,s)\, g \ , \quad g\Big|_{\tau=0} = u \ .$$

Differentiating with respect to t , we obtain

$$\frac{d}{d\tau}\left(\frac{\partial g}{\partial t}(\tau)\right) = \left(Z \frac{\partial g}{\partial t}\right)(\tau) + \left(\frac{\partial Z}{\partial t}\, g\right)(\tau) \ , \quad \frac{\partial g}{\partial t}\Big|_{\tau=0} = 0$$

Hence

$$\frac{\partial g}{\partial t}(\tau) = \int_0^\tau e^{(\tau-\mu)Z} \left(\frac{\partial Z}{\partial t}\, g\right)(\mu)\, d\mu$$

$$= \int_0^\tau e^{(\tau-\mu)Z} \frac{\partial Z}{\partial t} e^{\mu Z} u \, d\mu$$

$$= \int_0^\tau e^{(\tau-\mu)Z} \frac{\partial Z}{\partial t} e^{-(\tau-\mu)Z} d\mu \, g(\tau) \ .$$

Since it follows from (4.7) that Z and $\partial Z/\partial t$ satisfy (4.5) with $X = Z$, $Y = \partial Z/\partial t$ and $\ell = 3$, the application of Lemma 4.1 to the above integrand shows that

$$\frac{\partial g}{\partial t}(\tau) = \int_0^\tau \sum_{j=0}^{2} \frac{(\tau-\mu)^j}{j!} (\mathrm{ad}\ Z)^j \ (\partial Z/\partial t) \, d\mu \, g(\tau)$$

$$= \sum_{j=0}^{2} \frac{\tau^{j+1}}{(j+1)!} (\mathrm{ad}\ Z)^j \ (\partial Z/\partial t) \, g(\tau) \ .$$

Noting that $\sum_j (\mathrm{ad}\ Z)^j (\partial Z/\partial t)/(j+1)! = X$ and $g(1,0,s) = e^{sY} u$, we see that $g(1,t,s,x)$ satisfies (4.9). Hence, the uniqueness theorem of solutions for differential equations completes the proof.

Repeated application of this lemma gives

Corollary 4.3. *Let $X, Y \in T(\Omega)$ satisfy (4.7). For real σ_j ($j = 1,\ldots,4$) put*

$$(4.10) \quad Z(\sigma) = (\sigma_1 + \sigma_3)X + (\sigma_2 + \sigma_4)Y + \alpha(\sigma)[X,Y] + \beta(\sigma)[X,[X,Y]] + \gamma(\sigma)[Y,[Y,X]] \ ,$$

where

$$\alpha(\sigma) = (\sigma_1(\sigma_2 + \sigma_4) - \sigma_3(\sigma_2 - \sigma_4))/2 \ ,$$

$$\beta(\sigma) = ((\sigma_1 + \sigma_3)^2(\sigma_2 + \sigma_4) - 6\sigma_1\sigma_2\sigma_3)/12$$

$$\gamma(\sigma) = ((\sigma_1 + \sigma_3)(\sigma_2 + \sigma_4)^2 - 6\sigma_2\sigma_3\sigma_4)/12 \ .$$

If the transformation $X^{\sigma_1} Y^{\sigma_2} X^{\sigma_3} Y^{\sigma_4}$ from K to Ω is well-defined

for $\sum_{j=1}^{4} |\sigma_j| < \sigma_0$ ($\sigma_0 > 0$) *then we have*

$$e^{\sigma_1 X} e^{\sigma_2 Y} e^{\sigma_3 X} e^{\sigma_4 Y} u = e^{Z(\sigma)} u \ , \quad u \in C_0^\infty(K) \ , \tag{4.11}$$

provided that the transformation $Z(\sigma)^\tau$ *from* K *to* Ω *is well-defined for* $|\tau| < \tau_0$ ($\tau_0 > 1$).

For the compact set K of Lemma 4.2 (or Corollary 4.3) one can find a compact set K_1 of Ω such that $K \subsetneq K_1$ and the hypotheses of Lemma 4.2 (or Corollary 4.3) are fulfilled with replaced K by K_1 . If we take as $u \in C_0^\infty(K_1)$ the functions which equal the coordinates of Ω on K, we have two corollaries.

Corollary 4.4. Let the hypotheses of Lemma 4.2 be satisfied. Then we have for $|t| + |s| < t_0$

$$X^t Y^s x = Z(t,s)^1 x \quad , \quad x \in K. \tag{4.12}$$

Corollary 4.5. Let the hypotheses of Corollary 4.3 be satisfied. Then we have for $\sum_{j=1}^{4} |\sigma_j| < \sigma_0$

$$X^{\sigma_1} Y^{\sigma_2} X^{\sigma_3} Y^{\sigma_4} x = Z(\sigma)^1 x \ , \quad x \in K \ . \tag{4.13}$$

Proposition 4.6. For any positive σ_j ($j = 1,\dots,4$) *and any non-positive* a *there exist positives* $s_j = s_j(\sigma,a)$ ($j = 1,\dots,4$) *such that*

$$s_1 + s_3 = \sigma_2 + \sigma_4 \ , \quad s_2 + s_4 = \sigma_1 + \sigma_3 \tag{4.14}$$

$$\alpha(s) = -\alpha(\sigma) \tag{4.15}$$

$$a\beta(s) + \gamma(s) = \beta(\sigma) + a\gamma(\sigma) \ , \tag{4.16}$$

where α, β *and* γ *are the same as in Corollary 4.3. The mapping* $\sigma \to s(\sigma,a)$ *is bijective from* $\{\sigma_j > 0\ ;\ j = 1,\dots,4\ ,\ \sum_j \sigma_j < \sigma_0\}$ *to itself, where* σ_0 *is a fixed positive. Furthermore,* $s_j(\sigma,a)$ *are differentiable with respect to* $a \in (-\infty,0]$.

Proof. By (4.14), (4.16) is reduced to

(4.16)' $$s_2s_3(as_1 + s_4) = \sigma_2\sigma_3(\sigma_1 + a\sigma_4)$$

Noting $\alpha(s) = (s_1 + s_3)(s_2 + s_4) - 2s_2s_3$, in view of (4.14) it follows from (4.15) that

(4.15)' $$s_2s_3 = \sigma_1(\sigma_2 + \sigma_4) + \sigma_3\sigma_4 \quad (= h(\sigma)).$$

Therefore

$$as_1 + s_4 = \sigma_2\sigma_3(\sigma_1 + a\sigma_4)/h(\sigma).$$

Eliminating s_1 and s_4 in the left hand side by (4.14), we get

(4.17) $$s_2 + as_3 = ((\sigma_1+\sigma_3)^2\sigma_4 + \sigma_1^2\sigma_2 + a(\sigma_1(\sigma_2+\sigma_4)^2 + \sigma_3\sigma_4^2))/h(\sigma).$$

If we consider the intersection point of the hyperbola (4.15)' and the straight line (4.17) in (s_2,s_3) plane, it is easy to see that the simultaneous system of (4.15)' and (4.17) has the solution $(s_2(\sigma,a), s_3(\sigma,a))$ satisfying $0 < s_2 < \sigma_1 + \sigma_3$ and $0 < s_3 < \sigma_2 + \sigma_4$. It is easy to check the bijectivity and the differentiability of $s(\sigma,a)$.

Lemma 4.7. Let $X, Y \in T(\Omega)$ *satisfy the same assumptions as in Corollary 4.3 and moreover satisfy those assumptions with* X *and* Y *exchanged. Assume that we have*

(4.18) $[Y,[Y,X]] = a[X,[X,Y]] + W$ *in* $T(\Omega)$,

for some $a(x) \in C^\infty(\Omega)$ *and* $W \in T(\Omega)$ *satisfying*

(4.19) $Xa = Ya = Wa = 0$, $a \leqq 0$

(4.20) $[W,X] = [W,Y] = 0$.

For $x_0 \in \Sigma_0 = \{ x \in K ; W(x) = 0 \}$ *and for* $\sigma_j > 0$ $(j = 1,\dots,4,$ $\sum_j^4 \sigma_j < \sigma_0)$, *if we put* $\tilde{\sigma}_j = s_j(\sigma,a(x_0)) > 0$ $(j = 1,..,4)$ *for* $s_j(\cdot,\cdot)$ *given in Proposition 4.6 then we get*

(4.21) $X^{\sigma_1} Y^{\sigma_2} X^{\sigma_3} Y^{\sigma_4} x_0 = Y^{\tilde{\sigma}_1} X^{\tilde{\sigma}_2} Y^{\tilde{\sigma}_3} X^{\tilde{\sigma}_4} x_0$, $x_0 \in \Sigma_0$.

We also have (4.21) with X *and* Y *exchanged.*

Proof. We devide the proof into three steps

i) When $\Sigma \sigma_j < \sigma_0$ it follows from (4.14) that $\Sigma s_j(\sigma,a) < \sigma_0$. Hence the right hand side of (4.21) is well-defined. Since it follows from (4.19) that $Xs_j = 0$, we have

$$X^{s_j(\sigma,a(x_0))} x_0 = (s_j(\sigma,a(\cdot))X)^1 x_0 \quad , \quad x_0 \in K .$$

In view of this and the one with replaced X by Y, we obtain for $\tilde{\sigma}_j = s_j(\sigma,a(x_0))$

(4.22) $Y^{\tilde{\sigma}_1} X^{\tilde{\sigma}_2} Y^{\tilde{\sigma}_3} X^{\tilde{\sigma}_4} x_0 = (s_1 Y)^1 (s_2 X)^1 (s_3 Y)^1 (s_4 X)^1 x_0$, $x_0 \in K$,

because $a(x)$ is constant on the flows by X and Y in view of (4.19).

ii) Since $Xs_j = Ys_j = 0$, we can apply Lemma 4.2 to s_1X , s_2Y,... as if $s_j(\sigma,a)$ are constants. Hence, by Corollary 4.5 we obtain

$$(4.23)\quad (s_1Y)^1 (s_2X)^1 (s_3Y)^1 (s_4X)^1 x_0 = (\hat{Z}(s))^1 x_0 , \quad x_0 \in K ,$$

where $\hat{Z}(s)$ is the right hand side of (4.10) with replaced X, Y and σ_j by Y, X and s_j, respectively. By (4.18) we get

$$(4.24)\quad \hat{Z}(s) = (s_2 + s_4)X + (s_1 + s_3)Y - \alpha(s)[X,Y] + (a\beta(s) + \gamma(s))[X,[X,Y]] + \beta(s)W \equiv \hat{Z}_0(s) + \beta(s)W .$$

Similarly, by Corollary 4.5 we have

$$(4.25)\quad X^{\sigma_1} Y^{\sigma_2} X^{\sigma_3} Y^{\sigma_4} x_0 = Z(\sigma)^1 x_0 , \quad x_0 \in K ,$$

and from (4.18) we have

$$(4.26)\quad Z(\sigma) = Z_0(\sigma) + \gamma(\sigma)W ,$$

where

$$Z_0(\sigma) = (\sigma_1 + \sigma_3)X + (\sigma_2 + \sigma_4)Y + \alpha(\sigma)[X,Y] + (\beta(\sigma) + a\gamma(\sigma))[X,[X,Y]].$$

Since $s_j(\sigma,a)$ satisfy (4.14)-(4.16) we have

$$(4.27)\quad Z_0(\sigma) = \hat{Z}_0(s) \quad \text{in} \quad T(\Omega) .$$

iii) By means of (4.19) and (4.20) it easy to see that $[\tilde{Z}_0(s),\beta(s)W] = 0$. It follows from Corollary 4.4 that

$$(4.28)\quad \hat{Z}(s)^1 x_0 = \hat{Z}_0(s)^1 (\beta(s)W)^1 x_0 , \quad x_0 \in \Sigma_0 .$$

Since the similar equation for $Z(\sigma)$ holds , conjunction of

(4.22), (4.23), (4.25) and (4.27) gives (4.21) if we show

$$(4.29)\qquad (\beta(s)W)^{1}x_0 = x_0 = (\gamma(\sigma)W)^{1}x_0 \quad , \quad x_0 \in \Sigma_0 .$$

Putting $f(t,x) = (\beta(s)W)^{t}x$, f satisfies

$$\frac{df}{dt} = \beta(s)Wf \quad , \quad f(0,x) = x .$$

In view of $W(x_0) = 0$, the first equality of (4.29) follows from the uniqueness theorem of solutions for differential equations. Similarly we see the second equality of (4.29). The last assertion follows from the bijectivity of the mapping $\sigma \to s(\sigma,a)$.

Lemma 4.8. *Let $X, Y \in T(\Omega)$ satisfy*

$$(4.30)\qquad [X,[X,Y]] = [Y,[Y,X]] = 0 \quad \text{in} \quad T(\Omega) .$$

For a compact K of Ω let transformation $X^{\sigma_1}Y^{\sigma_2}X^{\sigma_3}$ and $Y^{\sigma_1}X^{\sigma_2}Y^{\sigma_3}$ from K of Ω be well-defined when real σ_j ($j = 1,2,3$) vary $\sum_{j}^{3} |\sigma_j| < \sigma_0$ for a fixed $\sigma_0 > 0$. Put

$$(4.31)\qquad V(\sigma) = (\sigma_1 + \sigma_3)X + \sigma_2 Y + \frac{1}{2}\sigma_2(\sigma_1 - \sigma_3)[X,Y]$$

and let $\tilde{V}(\sigma)$ equal the right hand side of (4.31) with X and Y exchanged. Assume that, if $\sum_{j}^{3} |\sigma_j| < \sigma_0$, transformations $V(\sigma)^{\tau}$ and $\tilde{V}(\sigma)^{\tau}$ from K to Ω are well-defined for $|\tau| < \tau_0$ ($\tau_0 > 1$). If for $s_j > 0$ ($j = 1,2,3$, $\sum_{j}^{3} s_j < \sigma_0$) we put

$$(4.32)\qquad \tilde{s}_1 = \frac{s_2 s_3}{s_1 + s_3} \quad , \quad \tilde{s}_2 = s_1 + s_3 \, , \, \tilde{s}_3 = \frac{s_1 s_2}{s_1 + s_3}$$

then we have

$$(4.33) \qquad X^{s_1}Y^{s_2}X^{s_3}x = Y^{\tilde{s}_1}X^{\tilde{s}_2}Y^{\tilde{s}_3}x \quad , \quad x \in K .$$

Proof. The application of Lemma 4.2 shows that the left and the right hand sides of (4.33) equal $V(s)^1x$ and $\hat{V}(\tilde{s})^1x$, respectively. Since it follows from (4.32) that $V(s) = \hat{V}(\tilde{s})$ in $T(\Omega)$, we get (4.33).

§5. Proof of Theorem 2.

In this section and the next section let λ_1 and λ_2 be equal to Example 2 in §1 with $b = 4$, that is,

$$(5.1) \qquad \lambda_1 = 0 \ , \ \lambda_2 = (t^2 - 4x_1t)\xi_2 + \xi_1 \ .$$

For a fixed $t_0 \in (0,T)$ and a fixed positive integer μ put

$$(5.2) \qquad \rho_0 = (0,\eta_0) \ , \ \eta_0 = (-\beta^2,1,\eta_0'') \ , \ \beta = t_0/2\mu.$$

Then, by Lemmma 2.2 we have

Proposition5.1. Admissible trajectories, issuing from ρ_0, are only $\mu+2$ trajectories, that is, two bicharacteristic curves of λ_1, λ_2, and trajectories $\tilde{C}(J_r, tt_r, \rho_0)$ of step $r = 1,\dots,\mu$. Here $J_r = (1,2,1,\dots,1)$ or $(2,1,2,\dots,1)$ according to r even or odd, and $tt_r = (t_1^r,\dots,t_r^r) = (\sigma_r,\dots,\sigma_1)$, $\sigma_q = (2q - 1)\beta$, $q = 1,\dots,\mu$. Furthermore, end points (at $t = t_0$) of those admissible trajectories are different each other. (See Fig. 9 and 10).

Proof. Since $H_1^2 p_2(0,\rho_0) = H_2^2 p_1(0,\rho_0) = 2$ and $(\lambda_1 - \lambda_2)(0,\rho_0) = \beta^2$, it follows from Lemma 2.2 that we have

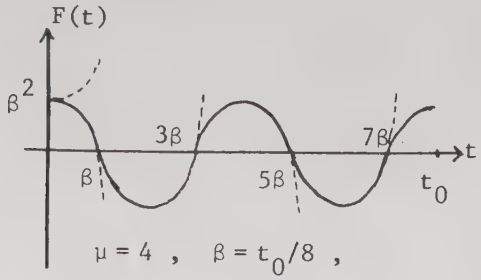


FIG. 9

$F_-(t) \equiv F(t;(1),\emptyset,\rho_0) = \beta^2 - t^2$, $F_+(t) \equiv F(t;(2),\emptyset,\rho_0) = \beta^2 + t^2$,

(where $C((1),\emptyset,\rho_0)$ and $C((2),\emptyset,\rho_0)$ denote two bicharacteristic curves of λ_1 and λ_2) and for $r = 1,\ldots,\mu$ we have

$$F_r(t) \equiv F(t;J_r, \tilde{t}_r,\rho_0) = (-1)^{r-q}(\beta^2 - (t - 2(r-q)\beta)^2)$$

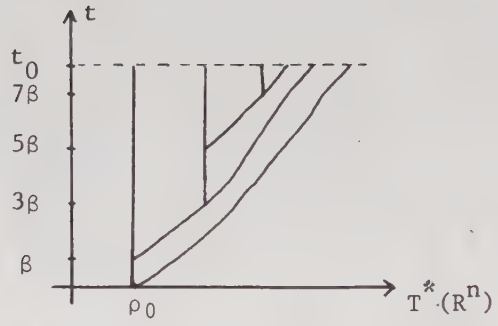
if $t \in [t^r_q, t^r_{q-1}]$ ($q = 1,\ldots,r+1$, $t^r_{r+1} = 0$, $t^r_0 = 0$).
(See Fig. 9). Hence, it is easy to check the first assertion. Noting that $F_{\pm}(t_0)$ and $F_r(t_0)$ are different, we see the last assertion.

The former assertion of Theorem 2 is obvious from this proposition. Let Δ_ν denote the closure of Δ^0_ν , that is,

$$\Delta_\nu = \{\tilde{t}_\nu = (t_1,\ldots,t_\nu) \ ; \ t_0 \geq t_1 \geq \cdots \geq t_\nu \geq 0 \ \}.$$

For the sake of convenience in the following argument we replace $\tilde{t}_\nu \in \Delta^0_\nu$ by $\tilde{t}_\nu \in \Delta_\nu$ in the definition of the trajectory.

Proposition 5.2. *If the trajectory* $C(J_\nu, \tilde{t}_\nu, \rho_0)$ *of*



$\mu = 4$, $\beta = t_0/8$, $J_4 = (1,2,1,2,1)$.

FIG. 10

step $\nu \geqq 1$ *is* ε*-admissible* ($\varepsilon << \beta^2$) *then it is close to one of admissible trajectories* $\tilde{C}(J_r, tt_r, \rho_0)$ ($r = 0,1,\ldots,\mu$) *given in Proposition 5.1 (,where* $\tilde{C}(J_0, tt_0, \rho_0) = C((1),\emptyset,\rho_0)$), *and moreover there exists a positive constant* c_0 *independent of* ν *such that we have either*

$$(5.3)\quad \begin{cases} \sigma_r + c_0\varepsilon \geqq t_1 \geqq t_2 \geqq \cdots \geqq t_{q_1} \geqq \sigma_r - c_0\varepsilon \\ \sigma_{r-1} + c_0\varepsilon \geqq t_{q_1+1} \geqq \cdots \geqq t_{q_1+q_2} \geqq \sigma_{r-1} - c_0\varepsilon \\ \qquad\qquad \cdots\cdots \\ \sigma_1 + c_0\varepsilon \geqq t_{q_1+\cdots q_{r-1}+1} \geqq \cdots \geqq t_{q_1+\cdots+q_r} \geqq \sigma_1 - c_0\varepsilon \end{cases}$$

for some odd $q_j = q_j(r) \geqq 1$ ($j = 1,\ldots,r$, $\sum_{j=1}^{r} q_j = \nu$), *or*

$$(5.3)'\quad \begin{cases} \sigma_{r+1} + c_0\varepsilon \geqq t_1 \geqq t_2 \geqq \cdots \geqq t_{q_0} \geqq \sigma_{r+1} - c_0\varepsilon \\ \sigma_r + c_0\varepsilon \geqq t_{q_0+1} \geqq \cdots \geqq t_{q_0+q_1} \geqq \sigma_r - c_0\varepsilon \\ \qquad\qquad \cdots \\ \sigma_1 + c_0\varepsilon \geqq t_{q_0+\cdots+q_{r-1}+1} \geqq \cdots \geqq t_{q_0+\cdots+q_r} \geqq \sigma_1 + c_0\varepsilon \end{cases}$$

for some even $q_0 = q_0(r)$ *and some odd* $q_j = q_j(r) \geq 1$ ($j = 1,\ldots,r$) *with* $\sum_{j=0}^{r} q_j = \nu$. *Here, if* $r = \mu$ *then we only have (5.3). If* $r = 0$ *then we only have (5.3)' with convention* $q_j = 0$ ($j = 1,\ldots,r$). *Therefore, if a trajectory* $C(J_\nu, \tilde{t}_\nu, \rho_0)$ *is close to the admissible trajectory* $\tilde{C}(J_\mu, \tilde{t}_\mu, \rho_0)$ *then its step* ν *is not smaller than* μ *and even or odd according to* μ *even or odd, and moreover* $J_\nu = (1,2,1,\ldots,1)$ *or* $J_\nu = (2,1,2,\ldots,1)$ *according to* μ *.even or odd.*

Proof. It is clear that if the trajectory $C(J_\nu, \tilde{t}_\nu, \rho_0)$ is ε-admissible then $F(t, J_\nu, \tilde{t}_\nu, \rho_0)$ is close to one of $F_-(t)$ and $F_r(t)$ given in the proof of Proposition 5.1. For $J_\nu \in \Pi_\nu^0$ and $\tilde{t}_\nu \in \Delta_\nu$, put $J'_{\nu-2} = (j_1,\ldots,j_q, j_{q+3},\ldots,j_{\nu+1})$ and $\tilde{t}'_{\nu-2} = (t_1,\ldots,t_{q-1},t_{q+2},\ldots,t_\nu)$. Then we have $C(J_\nu, \tilde{t}_\nu, \rho_0) = C(J'_{\nu-2}, \tilde{t}'_{\nu-2}, \rho_0)$ if $t_q = t_{q+1}$. Hence, it is easy to see that (5.3) or (5.3)' holds when $\varepsilon = 0$. Noting that $dF_r/dt = \pm 1$ at $t = t_q^r$, it is not difficult to see (5.3) or (5.3)' when $\varepsilon > 0$. The last assertion is obvious from (5.3) because q_j are odd.

In view of Proposition 5.2, the argument similar to Case 3 in §3 shows that if $U(t)$ is the solution of (C.P) for an initial value G satisfying (1.9) with $y_0 = 0$ and $\eta_0 = (-\beta^2, 1, \eta_0'')$ in (5.2) then WF $U(t_0)$ consists of, at most, $\mu+2$ end points of admissible trajectories given in Propsition 5.1.

From now on we shall show (1.10). As in [17], the initial value G of (C.P) is chosen to be a modification of the function given by Hörmander (see Example 2.3 of [9]), that is, $G = {}^t(g,g)$,

$$(5.4) \qquad g(x) = \sum_{k=1}^{\infty} k^{-2}\phi(k^3x)\exp ik^5x\cdot\eta_0 ,$$

where $\phi(x) \in C_0^\infty$ and $\hat{\phi}(0) = 1$. Then we have

$$WF\ G = \{(0,c\eta_0)\ ;\ c > 0\}.$$

In fact, since evry term is in C^∞ and the supports shrink to 0, we have sing supp $g = \{0\}$. Now

$$\hat{g}(\xi) = \sum_{k=1}^{\infty} k^{-3n-2}\hat{\phi}((\xi-k^5\eta_0)/k^3).$$

Noting that

$$(5.5) \qquad |k_0^5 - k^5|/k^3 \geqq 3k_0 , \quad k \neq k_0 ,$$

for any $N > 0$ we have for some $C_N > 0$

$$|\hat{\phi}((k_0^5 - k^5)\eta_0/k^3)| \leq C_N k_0^{-N} , \qquad k \neq k_0 .$$

Hence we get

$$|\hat{g}(k_0^5\eta_0)| \geqq k_0^{-3n-2}\hat{\phi}(0)/2 \quad \text{for large } k_0$$

and so $WF\ g \supset \{(0,c\eta_0)\ ;\ c > 0\}$. As in Example 2.3 of [9], we see $WF\ g \not\ni (0,\xi)$ if $\xi \notin \{c\eta_0\ ;\ c > 0\}$.

Let π denote the natural projection from $T^*(R^n)$ to R^n. Since λ_j are linear in ξ we can define a transformation $\pi_j(s,t) = \pi\chi_j(s,t)\pi^{-1}$ from R^n to R^n. Put $I_j(t,s)f = f(\pi_j(s,t)x)$ for $f \in \mathcal{E}'$ and put $I(t,s) = \begin{pmatrix} I_1(t,s) & 0 \\ 0 & I_2(t,s) \end{pmatrix}$. Then, as in [12] [13] we have the fundamental solution for L_0 of the form

$$E(t,s) = I(t,s) + \int_s^t I(t,\theta)W(\theta,s)d\theta$$

$$W(t,s) = \sum_{\nu=1}^{\infty} W_\nu(t,s) , \quad W_1(t,s) = \begin{pmatrix} 0 & I_2(t,s) \\ I_1(t,s) & 0 \end{pmatrix}$$

$$W_{\nu+1}(t,s) = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} W_1(t,t_1)W_1(t_1,t_2)\cdots W_1(t_\nu,s)$$

$$dt_\nu \cdots dt_2 dt_1 , \quad \nu \geq 1 .$$

Therefore, the solution $U(t_0,x) = {}^t(u_1(t_0,x),u_2(t_0,x))$ is represented in the form

$$(5.6) \qquad u_j(t_0,x) = g(\pi_j(t_0,0)x) + \sum_{\nu=1}^{\infty} \int_0^{t_0} \int_0^{t_1} \cdots \int_0^{t_{\nu-1}}$$

$$g(\pi_{J_\nu^j}(t_\nu)x)dt_\nu \cdots dt_1 ,$$

$$J_\nu^1 = (1,2,1,\ldots), \quad J_\nu^2 = (2,1,2,\ldots),$$

$$\pi_{J_\nu}(t_\nu) = \pi_{j_{\nu+1}}(0,t_\nu)\pi_{j_\nu}(t_\nu,t_{\nu-1})\cdots\pi_{j_2}(t_2,t_1)\pi_{j_1}(t_1,t_0).$$

Let $\delta_0 = (x_0,\xi_0)$ denote the end point of admissible trajectory of step μ given in Proposition 5.1. We can show $\delta_0 \in$ WF $u_1(t_0)$ or WF $u_2(t_0)$ according to μ even or odd. For the brevity, we assume that μ is even. For any neighborhood V of x_0, take $\psi(x) \in C_0^\infty(V)$ such that $\psi = 1$ in some neighborhood of x_0. We shall show for large integer k_0

$$(5.7) \qquad |\hat{\psi u}_1(t_0,k_0^5\xi_0)| \geq c_0 k_0^{-2-3n-(5/2)\mu} , \ (c_0 > 0),$$

where $\hat{\psi u}_1(t_0,\cdot)$ denotes the Fourier transform of ψu_1 with respect to variables x. Note that WF $u_1(t_0)$ consists of, at most, $\mu+2$ end points of admissible trajectories given in Propo-

sition 5.1. Since it follows from direct calculation (see (5.9) below) that ξ variables of end points are not collinear, we obtain for any N

$$(1-\hat{\psi})u_1(t_0,k_0^5\xi_0) = O(k_0^{-N}) \quad , \quad k_0 \to \infty \quad .$$

Hence, to prove (5.7) it suffices to show

$$(5.8) \qquad |\hat{u}_1(t_0,k_0^5\xi_0)| \geqq c_0' k_0^{-2-3n-(5/2)\mu} \quad , \ (c_0' > 0).$$

By (5.4) and (5.6) we have

$$\hat{u}_1(t_0,k_0^5\xi_0) = \sum_{k=1}^{\infty} k^{-2} \Big(w_0(k) + \sum_{\nu=1}^{\infty} \int_0^{t_0}\int_0^{t_1}\cdots\int_0^{t_{\nu-1}} w_\nu(k, tt_\nu)dt_\nu\cdots dt_1 \Big),$$

where

$$w_\nu(k, tt_\nu) = \int (\exp - i(k_0^5 x\cdot\xi_0 - k^5 \pi_{J_\nu^1}(tt_\nu)x\cdot\eta_0))\phi(k^3 \pi_{J_\nu^1}(tt_\nu)x)dx.$$

By the direct calculation we see for $\nu \geqq 1$

$$(5.9) \qquad \chi_{J_\nu^1}(tt_\nu)(x,\xi) \equiv \chi_1(t_0,t_1)\chi_2(t_1,t_2)\cdots\chi_{j_{\nu+1}}(t_\nu,0)(x,\xi)$$

$$= (x_1+A_\nu,\ x_2-B_\nu x_1+C_\nu,\ x'',\ \xi_1+B_\nu\xi_2,\ \xi'),$$

where

$$A_\nu = \sum_{j=1}^{\nu}(-1)^{j-1}t_j \quad , \quad B_\nu = 2\sum_{j=1}^{\nu}(-1)^{j-1}t_j^2 \quad ,$$

$$C_\nu = -\sum_{j=1}^{\nu} t_j^3 + 2\sum_{j=1}^{\nu}(-1)^{j-1}t_j^2 \sum_{m>j}^{\nu}(-1)^m t_m \quad , \ \nu \geqq 1 \ .$$

Since $\pi_{J_\nu^1}(tt_\nu) = \pi\chi_{J_\nu^1}(tt_\nu)^{-1}\pi^{-1}$, the change of variables $\pi_{J_\nu^1}x$ to x gives for $\nu \geqq 0$

$$w_\nu(k, tt_\nu) = k^{-3n}(\exp -ik_0^5(A_\nu\xi_{01} + C_\nu\xi_{02}))\times$$

$$\hat{\phi}((k_0^5(\xi_{01} - B_\nu\xi_{02}) - k^5\eta_{01})/k^3, (k_0^5\xi_0' - k^5\eta_0')/k^3)$$

with convention $A_\nu = B_\nu = C_\nu = 0$ if $\nu = 0$. Since we have $\xi_0' = \eta_0'$ by (5.9), it follows from (5.5) that if $k \neq k_0$ then we get for any N

$$w_\nu(k, \tau_\nu) = O(k_0^{-N}) \quad , \quad k_0 \to \infty \ .$$

Since we have also $\xi_{01} - \eta_{01} = t_0^2$ by (5.9), we see

$$w_0(k_0) = O(k_0^{-N}) \quad , \quad k_0 \to \infty \ .$$

Hence we may only consider the integral

$$(5.10) \qquad I_\nu(k_0) \equiv \int_0^{t_0} \cdots \int_0^{t_{\nu-1}} w_\nu(k_0, \tau_\nu) dt_\nu \cdots dt_1 \times k_0^{3n}$$

$$= \int_{\Delta_\nu} e^{ik_0^5 F_\nu(\tau_\nu)} G_\nu(\tau_\nu, k_0) d\tau_\nu \quad , \qquad \nu \geq 1 \ ,$$

where $F_\nu(\tau_\nu) = A_\nu(\beta^2 - t_0^2) - C_\nu$, $G_\nu(\tau_\nu, k_0) = \hat{\phi}(k_0^2(t_0^2 - B_\nu), 0)$. To prove (5.8), that is, to complete the proof of Theorem 2 it suffices to show

Lemma 5.3. *If ν is odd or $\nu < \mu$, then for any positive integer N there exists a constant C_N independent of ν such that*

$$(5.11) \qquad |I_\nu(k_0)| \leq C_N^\nu k_0^{-N}/(\nu-2N)! \quad \text{for large } k_0 \ .$$

If ν is even and $\nu \geq \mu + 2$, there exists a constant C_1 independent of ν such that

(5.12) $|I_\nu(k_0)| \leq C_1^\nu k_0^{-(5/2)\mu - (\nu-\mu)}/[\nu/\mu]!$ *for large* k_0 .

Furthermore we get for some $c_0 > 0$

(5.13) $|I_\mu(k_0)| \geq c_0 k_0^{-(5/2)\mu}$ *for large* k_0 .

§6. Proof of Lemma 5.3.

For the brevity, in this section we write k instead of k_0. For a function $h(t) \in C_0^\infty(R^1)$ such that $h = 1$ ($|t|<1/2$) and $h = 0$ ($|t|>1$) we put

(6.1) $$\psi_\nu(\tilde{t}_\nu) = h(\kappa^{-\nu} k (t_0^2 - B_\nu)) ,$$

where κ is a fixed small positive constant satisfying

(6.2) $$\kappa^{-\nu} >> \nu^\mu .$$

Because the volume of Δ_ν equals $t_0^\nu/\nu!$ and because for any integer $N > 0$ there exists a constant C_N such that $|\hat{\phi}(\xi)| \leq C_N|\xi|^{-N}$ for large $|\xi|$, we have

(6.3) $$\left|\int_{\Delta_\nu} e^{ik^5 F_\nu}(1-\psi_\nu)G_\nu \, d\tilde{t}_\nu\right| \leq C_N t_0^\nu \kappa^{-\nu N} k^{-N}/\nu! .$$

To find the points $\tilde{t}_\nu \in \Delta_\nu$ satisfying

(6.4) $$\partial_{t_q} F_\nu = \partial_{t_q} A_\nu(\beta^2 - t_0^2) - \partial_{t_q} C_\nu = 0 , \quad q = 1,\ldots,\nu .$$

(6.5) $$t_0^2 - B_\nu = 0 ,$$

we state the following lemma, which is derived from Theorem 2.3 of Kumano-go-Taniguchi-Tozaki [12].

Lemma 6.1. *Let* $\phi_j(t,s) = \phi_j(t,s;x,\xi)$ $(j = 1,2)$ *be the solutions of eiconal equations*

$$\partial_t\phi_j + \lambda_j(t,x,d_x\phi_j) = 0 \quad , \quad \phi_j(t,s)_{t=s} = x\cdot\xi \ .$$

For a sequence of phase functions $\phi_{j_1}(t_0,t_1)$, $\phi_{j_2}(t_1,t_2),\dots,$ $\phi_{j_{\nu+1}}(t_\nu,0)$ $(J_\nu = (j_1,j_2,\dots,j_{\nu+1}),\ tt_\nu \in \Delta_\nu)$ *let* $\Phi_{J_\nu}(t_0, tt_\nu, 0;x,\xi)$ *denote the multi-product*

$$(\phi_{j_1}(t_0,t_1)\#\phi_{j_2}(t_1,t_2)\#\cdots\#\phi_{j_{\nu+1}}(t_\nu,0))(x,\xi) \ .$$

Then we have

$$\partial_{t_q}\Phi_{J_\nu}(t_0, tt_\nu, 0;x,\xi)\Big|_{\substack{x=z_0^\nu \\ \xi=\eta_0}} = \lambda_{j_q}(t_q,\rho_q) - \lambda_{j_{q+1}}(t_q,\rho_q) \ , \tag{6.6}$$

$$q = 1,2,\dots,\nu \ ,$$

where $\rho_q^\nu = \rho_q^\nu(tt_\nu) = \chi_{j_{q+1}}(t_q,t_{q+1})\cdots\chi_{j_{\nu+1}}(t_\nu,0)\rho_0$ *for* $q = 1,\dots,\nu$ *and* $z_0^\nu = \pi\chi_{j_1}(t_0,t_1)\cdots\chi_{j_{\nu+1}}(t_\nu,0)\rho_0$.

Since $\phi_j(t,s;x,\xi) = (\pi_j(s,t)x)\cdot\xi$ we get

$$\Phi_{J_\nu^1}(t_0, tt_\nu, 0;x,\xi) = (\pi_{J_\nu^1}(tt_\nu)x)\cdot\xi$$

$$= (x_1 - A_\nu)\xi_1 + (x_2 - C_\nu + (x_1 - A_\nu)B_\nu)\xi_2 + x''\cdot\xi'' \ .$$

By $z_{01}^\nu = A_\nu$ and $\eta_0 = (-\beta^2,1,\eta_0'')$ we have

$$\partial_{t_q}\Phi_{J_\nu^1}\Big|_{\substack{x=z_0^\nu \\ \xi=\eta_0}} = \partial_{t_q}A_\nu(\beta^2 - B_\nu) - \partial_{t_q}C_\nu$$

$$= \partial_{t_q}F_\nu + (-1)^q(B_\nu - t_0^2) \ . \tag{6.7}$$

Therefore, it follows from (6.6) that if a $\tilde{t}_\nu \in \Delta_\nu$ satisfies (6.4) and (6.5) then we have

$$(\lambda_1 - \lambda_2)(\tilde{t}_q, \rho_q(\tilde{t}_\nu)) = 0 , \quad q = 1,\dots,\nu,$$

that is, the trajectory $C(J^1_\nu, \tilde{t}_\nu, \rho_0)$ is admissible. By Proposition 5.2 with $\varepsilon = 0$ we see that if and only if ν is even and not smaller than μ, then there exist points $\tilde{t}^p_\nu \in \Delta_\nu$ ($p = 1,\dots,N_\nu$) satisfying (6.4) and (6.5). Furthermore, such a $\tilde{t}^p_\nu = (\tilde{t}_1,\dots,\tilde{t}_\nu)$ is given by

$$(6.8) \qquad \begin{cases} \tilde{t}_1 = \tilde{t}_2 = \cdots = \tilde{t}_{q_1} = \sigma_\mu \\ \tilde{t}_{q_1+1} = \cdots = \tilde{t}_{q_1+q_2} = \sigma_{\mu-1} \\ \qquad \cdots \\ \tilde{t}_{q_1+q_2+\cdots+q_{\mu-1}+1} = \cdots = \tilde{t}_{q_1+\cdots+q_\mu} = \sigma_1 \end{cases} ,$$

where $q_j \geq 1$ ($j = 1,\dots,\mu$) are odd and $\sum q_j = \nu$. In fact, if $\tilde{t}^p_\nu$ satisfies (6.4) then it is given by (5.3) or (5.3)' with $\varepsilon = 0$. Since it also satisfies (6.5) , we need $r = \mu$. The number of points $\tilde{t}^p_\nu$ is obviously bounded above by 2^ν .

In view of (6.3) we may only show Lemma 5.3 with $I_\nu(k)$ replaced by

$$(6.9) \qquad I^0_\nu(k) = \int_{\Delta_\nu} e^{ik^5 F_\nu} \psi_\nu G_\nu \, dt_\nu .$$

We consider the case for even $\nu \geq \mu$. Putting

$$(6.10) \qquad \phi^p_\nu(t_\nu) = h(\kappa^{-\nu}k|t_\nu - \tilde{t}^p_\nu|) , \quad p = 1,\dots,N_\nu$$

$$G^1_\nu(t_\nu,k) = \Big(1 - \sum_p \phi^p_\nu\Big)\psi_\nu G_\nu ,$$

we have

$$(6.11)\quad I_\nu^0(k) = \int_{\Delta_\nu} e^{ik^5 F_\nu} G_\nu^1 \, d\tau_\nu + \sum_p \int_{\Delta_\nu} e^{ik^5 F_\nu} \phi_\nu^p \psi_\nu G_\nu \, d\tau_\nu$$
$$\equiv I_\nu^1(k) + \sum_p I_\nu^{2,p}(k) ,$$

and for any multi-index α there exists a constant C_α such that

$$(6.12)\quad |\partial_{\tau_\nu}^\alpha G_\nu^1| \leq C_\alpha^\nu k^{2|\alpha|} .$$

Here and in what follows we denote different constants independent of ν (but depending α or N) by the same notation C_0 (C_α or C_N).

We shall show that for any integer $N > 0$ we get

$$(6.13)\quad |I_\nu^1(k)| \leq C_N^\nu k^{-2N}/(\nu - 2N)! .$$

In view of (6.8) and (6.10) it follws from Proposition 5.2 that

$$(6.14)\quad \begin{aligned} &\textit{for any } \tau_\nu \in \Delta_\nu \cap \textit{supp } G_\nu^1 \textit{ there exists a } q \ (\ 1 \leq q \leq \nu\ ) \\ &\textit{such that } |\partial_{t_q} F_\nu(\tau_\nu)| \geq c_1 \kappa^\nu/\nu k , \end{aligned}$$

where c_1 is a positive constant independent of ν . Taking a suitable sequence of points τ_ν^j ($j = 1,2,\dots$) in $\Delta_\nu \cap \text{supp } G_\nu^1$ and putting $h_j(\tau_\nu) = h(\kappa^{-2\nu} k|\tau_\nu - \tau_\nu^j|)$, we make the partition of unity over $\Delta_\nu \cap \text{supp } G_\nu^1$,

$$\sum_j \psi_\nu^j(\tau_\nu) \equiv 1 , \quad \psi_\nu^j = h_j / \sum_j h_j .$$

Obviously we have

$$(6.15)\quad |\partial_{\tau_\nu}^\alpha \psi_\nu^j| \leq C_\alpha^\nu k^{|\alpha|} .$$

In view of $\kappa^2 << \kappa$ it follows from (6.14) that for any ψ_ν^j we have for some q

$$|\partial_{t_q} F_\nu| \geq c_1 \kappa^\nu / 2\nu k \quad \text{on supp } \psi_\nu^j . \tag{6.16}$$

Note that

$$\int_{\Delta_\nu} \cdot \, d\mathrm{t}_\nu = \int_{\Delta_{\nu-1}} dt_1 \cdots dt_{q-1} dt_{q+1} \cdots dt_\nu \int_{t_{q+1}}^{t_{q-1}} \cdot \, dt_q . \tag{6.17}$$

By means of (6.16) it follows from integration by parts that

$$\begin{aligned} \int_{t_{q+1}}^{t_{q-1}} e^{ik^5 F_\nu} \psi_\nu^j G_\nu^1 dt_q &= [\, -ik^{-5} e^{ik^5 F_\nu} \psi_\nu^j G_\nu^1 (\partial_{t_q} F_\nu)^{-1}]_{t_q = t_{q+1}}^{t_q = t_{q-1}} \\ &\quad + ik^{-5} \int_{t_{q+1}}^{t_{q-1}} e^{ik^5 F_\nu} \partial_{t_q} (\psi_\nu^j G_\nu^1 (\partial_{t_q} F_\nu)^{-1}) dt_q . \end{aligned} \tag{6.18}$$

Hence, by (6.12),(6.15) and (6.16) we see that the left hand side of (6.18) is bounded above by $C_0 k^{-2}$. We may assume that the number of ψ_ν^j whose supports overlap each other is bounded above by 2^ν. Because the number of resolutions as in (6.17) is ν and because the volume of $\Delta_{\nu-1}$ is equal to $t_0^{\nu-1}/(\nu-1)!$, we obtain (6.13) with $N = 1$. It is easy to check

$$F_\nu(\mathrm{t}_\nu)\Big|_{t_q = t_{q+1}} = F_{\nu-2}(t_1, \ldots, t_{q-1}, t_{q+2}, \ldots, t_\nu) ,$$

so that we have

$$\begin{aligned} &\int_{\Delta_{\nu-1}} e^{ik^5 F_\nu} \psi_\nu^j G_\nu^1 (\partial_{t_q} F_\nu)^{-1} \Big|_{t_q = t_{q+1}} dt_1 \cdots \widehat{\ q\ } \cdots dt_\nu \\ &\qquad = \int_{\Delta_{\nu-2}} e^{ik^5 F_{\nu-2}} G_{\nu-2}^2 \, dt_1 \cdots \widehat{\ q, q+1\ } \cdots dt_\nu \end{aligned} \tag{6.19}$$

where

$$G^2_{\nu-2}(t_1,..,t_{q-1},t_{q+2},...,t_\nu) = \int_{t_{q+2}}^{t_{q-1}} \psi^j_\nu G^1_\nu (\partial_{t_q} F_\nu)^{-1} \Big|_{t_q=t_{q+1}} dt_{q+1}.$$

Noting that $t_{q-1} \geqq t_q \geqq t_{q+1} \geqq t_{q+2}$ if $tt_\nu \in \Delta_\nu$, we see that supp $G^2_{\nu-2} \cap \Delta_{\nu-2}$ is apart from the points $\tilde{tt}^p_{\nu-2}$ ($p = 1,...,N_{\nu-2}$) satisfying (6.8) with $\nu = \nu-2$. Hence, by means of Proposition 5.2 we get (6.14) with replaced G^1_ν and F_ν by $G^2_{\nu-2}$ and $F_{\nu-2}$. By taking again the partition of unity over $\Delta_{\nu-2} \cap$ supp $G^2_{\nu-2}$, we can repeat the proceding argument for the right hand side of (6.19). Repeating this procedure N times, we get (6.13). Because the volume of $\Delta_{\nu-2N}$ is $t_0^{\nu-2N}/(\nu-2N)!$ and because the number of overlaps of coverings and the number of resolutions of Δ_ν into $\Delta_{\nu-2N}$ and Δ_{2N} are bounded above by C_N^ν. Here we neglect the case for $\nu \leqq 2N$. In this case, after integration by parts $\nu/2$ times we obtain terms which are not integral. However, such terms equal zero because supp G^1_ν does not contain $(0,...,0)$ and $(t_0,...,t_0)$.

When ν is odd or smaller than μ, we can apply the preceding argument to $I^0_\nu(k)$ instead of $I^1_\nu(k)$ and so we get an estimate of the form (6.13) for $I^0_\nu(k)$. In fact we have (6.14) with replaced G^1_ν by $\psi_\nu G_\nu$. Therefore, in view of (6.3) we get (5.11) of Lemma 5.3. Now we may only show (5.12) and (5.13) with replaced $I_\nu(k)$ by $I^{2,p}_\nu(k)$ because the number N_ν of points $\tilde{tt}^p_\nu$ is bounded above by 2^ν.

We take a change of variables $tt_\nu \to ss_\nu = (ss_\mu, ss^1, ss^2, ..., ss^\mu)$ ($ss_\mu = (s_1,...,s_\mu)$, $ss^j = (s^j_1,...,s^j_{q_j-1})$, $j = 1,...,\mu$) as

follows;

$$
\begin{aligned}
&s_1^1 = t_1 - t_2,\ s_2^1 = t_2 - t_3, \dots,\ s_{q_1-1}^1 = t_{q_1+q_2-1} - t_{q_1+q_2},\\
&s_1 = (t_1 + \cdots + t_{q_1})/q_1\ ,\\
&s_1^2 = t_{q_1+1} - t_{q_1+2}\ , \dots,\ s_{q_2}^2 = t_{q_1+q_2-1} - t_{q_1+q_2}\ ,\\
&s_2 = (t_{q_1+1} + \cdots + t_{q_1+q_2})/q_2,\\
&\quad \cdots
\end{aligned}
\tag{6.20}
$$

Noting $\operatorname{supp} \phi_\nu^p$, we obtain

$$
I_\nu^{2,p}(k) = \int_{D_1} ds^1 \int_{D_2} ds^2 \cdots \int_{D_\mu} ds^\mu \int e^{ik\tilde{F}_\nu} \tilde{G}_\nu\, ds_\mu\ ,
\tag{6.21}
$$

where $D_j = \{(s_1^j,\dots,s_{q_j-1}^j)\ ;\ s_q^j \geq 0\ ,\ \sum_q s_q^j \leq \kappa^\nu(q_j-1)/k\ \}$,

$\tilde{F}_\nu(s_\nu) = F_\nu(t_\nu)$ and $\tilde{G}_\nu(s_\nu,k) = (\phi_\nu^p \psi_\nu G_\nu)(t_\nu,k)$. Regarding $(s^1, s^2,\dots, s^\mu)$ $(\equiv s'_\nu)$ as parameters, we shall apply the stationary phase method (see §1-2 of Duistermaat [3]) to

$$
I_\nu(k; s'_\nu) = \int e^{ik\tilde{F}_\nu(s_\mu, s'_\nu)}\, \tilde{G}_\nu(s_\mu, s'_\nu, k)\, ds_\mu\ .
\tag{6.22}
$$

Let $s_\mu(s'_\nu)$ denote a stationary point of $\tilde{F}_\nu(\cdot, s'_\nu)$, that is, $s_\mu(s'_\nu)$ satisfies

$$
\partial_{s_q} \tilde{F}_\nu(\cdot\ , s'_\nu) = 0 \quad (q = 1,\dots,\mu).
\tag{6.23}
$$

Noting that $\tilde{F}_\nu(s_\mu,0) = F_\mu(s_\mu)$, we see that $s_\mu(s'_\nu)$ is a continuous function with respect to $s'_\nu \in \prod_j D_j$ and $s_\mu(0)$ is equal to $(\sigma_\mu,\dots,\sigma_1)$. Put

$$
Q(s'_\nu) = \left(\partial_{s_q} \partial_{s_{q'}} \tilde{F}_\nu(s_\mu, s'_\nu)\ ;\ \begin{matrix} q = 1,\dots,\mu \\ q' = 1,\dots,\mu \end{matrix} \right)_{s_\mu = s_\mu(s'_\nu)}\ .
$$

It is not difficult to check

$$(6.24)\quad \det Q(0) = |\partial_{s_q}\partial_{s_q'}F_\mu(\sigma_\mu,\dots,\sigma_1)| = (1-4\mu^2)(t_0/\mu)^\mu .$$

So we may assume that when ss_ν' varies in ΠD_j there exists a positive constant c_2 independent of ν such that

$$(6.25)\qquad |\det Q(\mathrm{ss}_\nu')| \geqq c_2^\nu .$$

Since $\tilde{G}_\nu$ also satisfies an estimate of the form (6.12), the application of Proposition 1.2.4 of [3] to (6.22) gives

$$(6.26)\quad I_\nu(k,\mathrm{ss}_\nu') \sim (2\pi/k^5)^{\mu/2}\,|\det Q|^{-1/2}\,e^{(\pi i/4)\operatorname{sgn} Q} \times \sum_{j=0}^{\infty}(R^j\tilde{G}_\nu)\,k^{-5j}/j!\ ,\quad k\to\infty\ ,$$

where $Q = Q(\mathrm{ss}_\nu')$, $R = i\langle Q^{-1}\partial_{\tilde{\mathrm{ss}}_\mu},\partial_{\tilde{\mathrm{ss}}_\mu}\rangle/2$ and $\tilde{\tilde{G}}_\nu(\tilde{\mathrm{ss}}_\mu,\mathrm{ss}_\nu') = \tilde{G}_\nu(\mathrm{ss}_\mu,\mathrm{ss}_\nu')$. Here $(\mathrm{ss}_\mu,\mathrm{ss}_\nu') \to \tilde{\mathrm{ss}}_\mu$ is a C^∞ - mapping such that

$$(6.27)\quad \tilde{\mathrm{ss}}_\mu(\mathrm{ss}_\mu,\mathrm{ss}_\nu') = \mathrm{ss}_\mu - \mathrm{ss}_\mu(\mathrm{ss}_\nu') + O(|\mathrm{ss}_\mu - \mathrm{ss}_\mu(\mathrm{ss}_\nu')|^2)$$

$$(6.28)\quad \tilde{F}_\nu(\mathrm{ss}_\mu,\mathrm{ss}_\nu') = \tilde{F}_\nu(\mathrm{ss}_\mu(\mathrm{ss}_\nu'),\mathrm{ss}_\nu') + \langle Q\tilde{\mathrm{ss}}_\mu,\tilde{\mathrm{ss}}_\mu\rangle/2 .$$

Since $I_\mu^{2,1}(k)$ equals $I_\mu(k,0)$ we obtain (5.13) from (6.26). Since it follows from (6.2) that the volume of D_j is not bigger than $k^{(1-q_j)}/q_j!$, in view of (6.26) and (6.21) we get (5.12).

We refer results of Alinhac [1], Hanges [5] and Taniguchi=Tozaki [20] (cf. Example 1 of Ralston [19]) concerning the branching (or the precise propagation) of singularities for hyperbolic operators. Our method developed in §5 and 6 is quite different from theirs. It seems to be difficult to apply directly their methods to our example.

§7. Example 3 and concluding remark.

Let L_1 be a hyperbolic 2×2 system in $(-T,T)\times R^4$ of the form (0.1) whose $B = \sqrt{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, λ_1 and λ_2 are equal to Example 3 in §1, that is,

$$(7.1) \qquad \lambda_1 = \xi_1 + t\xi_2 , \quad \lambda_2 = \xi_1 + \xi_3 + (t^2 - bx_3 t)\xi_4 .$$

Let $U(t)$ be the solution of (C.P) with $L = L_1$ for the initial value G such that

$$(7.2) \qquad \text{WF } G = \{ (0,c\eta) \ ; \ c > 0 \}.$$

For suitable G satisfying (7.2) we get the following:

Case 1. If $b = 2$ and $\eta = (0,0,0,1)$ then

$$(7.3) \quad \text{WF } U(t_0) = \Gamma^1(t_0 \ ; \text{WF } G)$$

$$= \{(t_0,(t_0^2-\theta^2)/2,\theta,-\theta^3/3,0,0,c\theta^2,c) \ ; \ c > 0 ,$$

$$0 \leqq \theta \leqq t_0 \} \ . \ (\text{See Fig. 11}).$$

Case 2. If $b = 0$ and $\eta = (0,3,-2,1)$ then for $t_0 > 2$ we have

$$(7.4) \quad \text{WF } U(t_0) = \Gamma^2(t_0 \ ; \text{WF } G)$$

$$= \{ \text{ end points of } C(J_\nu, \mathfrak{tt}_\nu, (0,c\eta)) \ ; \ c > 0 ,$$

$$J_\nu = \text{resp. } (1),(2),(1,2),(1,2),(2,1),(2,1),$$

$$(1,2,1),(2,1,2) \ , \ \mathfrak{tt}_\nu = \text{resp. } \emptyset,\emptyset,(1),(2),$$

$$(1),(2),(1,2),(1,2)\} \ .$$

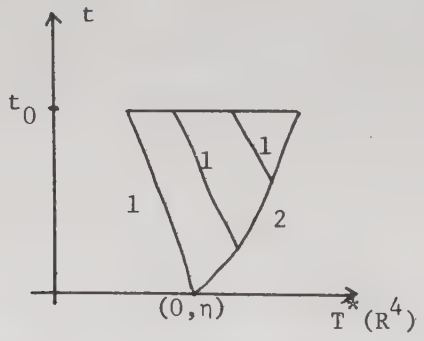


FIG. 11

Case 3. If $b \leq 2$ and $\eta = (1,0,0,0)$ then

$$(7.5)\quad WF\ U(t_0) = \tilde{\Gamma}^3(t_0\ ;\ WF\ G\ )$$

$$= \{(t_0,(t_0^2-t_1^2+t_2^2-t_3^2)/2,t_1-t_2+t_3,f(t),c\eta)\ ;$$

$$c > 0\ ,\ 0 \leq t_3 \leq t_2 \leq t_1 \leq t_0\ \},$$

where $f(t) = (1-b)(t_1^3-t_2^3+t_3^3)/3 + b(t_2-t_3)(t_1^2-t_2^2)/2$.
Thus, $WF\ U(t_0)$ is three dimension conic subset in $T^*(R^4)$.

Indeed, Corollary 3.2 and the argument of Case 1,2 in §3 show that the right hand sides of (7.3) and (7.4) contains the left hand sides, respectively. If we take a G of the form (5.4) then we obtain relations of converse inclusion in (7.3) and (7.4) by the same method as in §5 and 6. The application of Theorem of [17] to Case 3 shows that for another G satisfying (7.2) (see §2 of [17]) we have

$$WF\ U(t_0) \supset \tilde{\Gamma}(t_0\ ;\ WF\ G\ )\ ,$$

where $\tilde{\Gamma} = \bigcup_{k=0} \Gamma^k$. In view of $\tilde{\Gamma} \supset \tilde{\Gamma}^3$, Theorem 1 and Remark

after Lemma 3.1 give (7.5).

We remark that the closure of $\tilde{\Gamma}$ is not generally equal to Γ defined by (1.5). For instance, if $b = 4$ and $\eta = (0,0,0,1)$ then we see

$$\bar{\tilde{\Gamma}}(t_0 ; \mathrm{WF}\ G) \neq \Gamma(t_0 ; \mathrm{WF}\ G).$$

In fact, if $\eta_\mu = (0,0,-(t_0/2\mu)^2,1)$, $(\mu = 1,2,\ldots)$, and if δ_μ denotes the end point of admissible trajectory of step μ, issuing from $(0,\eta_\mu)$ then $\Gamma(t_0 ; \mathrm{WF}\ G)$ contains accumulating points δ_∞ of $\{\delta_\mu\}_{\mu=1}^{\infty}$. On the other hand, it is clear that $\bar{\tilde{\Gamma}}(t_0 ; \mathrm{WF}\ G) = \bar{\Gamma}^0(t_0 ; \mathrm{WF}\ G) = \Gamma^0(t_0 ; \mathrm{WF}\ G)$, and we get $\Gamma^0(t_0 ; \mathrm{WF}\ G) \not\ni \delta_\infty$ because $\lambda_1 - \lambda_2 = 0$ at (t_0,δ_∞). It is an open problem whether we have

(7.6) $\qquad \mathrm{WF}\ U(t_0) \ni \delta_\infty$.

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ON THE PROPAGATION OF THE WAVE FRONT SET OF A SOLUTION FOR A HYPERBOLIC SYSTEM

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Abstract. In this paper we consider the Cauchy problem for a hyperbolic system with characteristics of variable multiplicity and show that the wave front set of a certain solution propagates precisely not only along bicharacteristic curves but also on the set spanned by bicharacteristic curves. We take as the initial value of such a solution the function given by Hörmander [2], whose wave front set consists of only one point in the cotangent space.

Introduction. Chazarain [1] proved that, for hyperbolic equations with characteristics of constant multiplicity, wave front sets of solutions propagate along bicharacteristic curves, as in the strictly hyperbolic case, assuming the so-called Levi condition. (cf. Mizohata-Ohya [7]). On the other hand, for hyperbolic equations with characteristics of variable multiplicity, the propagation of wave front sets of solutions is more involved, even if equations can be reduced to symmetric systems. Kumano-go and Taniguchi [5] constructed a fundamental solution for a hyperbolic system of order 1 with the diagonal principal part by using Fourier integral operators with muti-phase. (See also [4]). Their fundamental solution shows that wave front sets of solutions may propagate not only along bicharacteristic curves but also along admissible trajectories, where we say that a curve obtained by linking bicharacteristic curves is an admissible trajectory if two characteristic roots of adjacent bicharacteristic curves coincide at each joint. However, their result does not show that wave front sets of solutions propagate precisely along admissible trajectories. It should be noted that the propagation along admissible trajectories depends on the lower order term of the system. Indeed, if the lower order term of the system is also diagonal, wave front sets of solutions propagate only along bicharacteristic curves.

The purpose of this paper is to show that, for a hyperbolic system, the wave front set of a certain solution of the Cauchy problem propagates precisely along all admissible trajectories. The initial value of such a solution is chosen to be the special form (see (8) in § 2) of Example 2.3 of Hörmander [2],

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$$\sum_{k=1}^{\infty} k^{-2}\phi(kx)\exp ik^2x\eta,\ \eta\neq 0$$

$$(\phi\in C_0^\infty,\ \hat{\phi}\geqq 0,\ \hat{\phi}(0)=1),$$

whose wave front set is $\{(0,c\eta);c>0\}$.

§1. **Notations and result.** Let L be a hyperbolic $\ell\times\ell$ system of the form

$$(1)\qquad L=\begin{pmatrix} D_t+\lambda_1(t,x,D_x) & & 0 \\ & \ddots & \\ 0 & & D_t+\lambda_\ell(t,x,D_x) \end{pmatrix}+iB(t,x')$$

on $[0,T]\times R_x^n$,

where $i=\sqrt{-1}$, $x=(x_1,x')$, $D_x=(D_1,\cdots,D_n)$, $D_t=-i\dfrac{\partial}{\partial t}$, and $D_j=-i\dfrac{\partial}{\partial x_j}$. Here

$$(2)\qquad \lambda_j(t,x,D_x)=a(t,x_1)D_1+\sum_{k=2}^{n}a_{jk}(t,x')D_k,\ j=1,\ldots,\ell,$$

$$(3)\qquad a,a_{jk}\in\mathscr{B}^\infty,\ \text{real valued.}$$

We assume that $B=(b_{ij}(t,x'))$ satisfies

$$(4)\qquad b_{ij}(t,x')>0,\ b_{ij}\in\mathscr{B}^\infty.$$

We consider the Cauchy problem

$$(\text{C.P})\qquad LU=0,\ U\mid_{t=0}=G,\ G\in\mathscr{E}'$$

A curve $\{(t,x(t),\xi(t));t\in[0,T]\}$ is called a bicharacteristic curve of $\lambda(t,x,\xi)$ if $dx/dt=d_\xi\lambda(t,x,\xi)$ and $d\xi/dt=-d_x\lambda(t,x,\xi)$. Let $\chi_j(t,s)$ $(t,s\in[0,T],\ j=1,\cdots,\ell)$ denote the transformation;

$$T^*(R_x^n)\ni\rho\to\chi_j(t,s)\rho\in T^*(R_x^n)$$

such that $\{(t,\chi_j(t,s)\rho;t\in[0,T]\}$ is the bicharacteristic curve of λ_j through (s,ρ). For an integer $\nu\geqq 0$ let $J(\nu)=(j(1),j(2),\cdots,j(\nu+1))$ be a $(\nu+1)$-repeated permutation of elements $\{1,2,\cdots,\ell\}$ and let Π_ν denote the set of all $J(\nu)$. Let Π_ν^0 denote the set of $J(\nu)$ whose any two adjacent components are different.

Let t_0 be any but a fixed point in $(0,T]$. A continuous curve $\{(t,x(t),\xi(t));t\in[0,t_0]\}$ is called a trajectory of step ν, issuing from ρ if it starts from ρ when $t=0$ and if, for some $J(\nu)\in\Pi_\nu^0$ and some $\{t_1,\cdots,t_\nu\}$ satisfying $t_0>t_1>\cdots>t_\nu>0$, it is the bicharacteristic curve of $\lambda_{j(q)}$ when $t\in[t_q,t_{q-1}]$ $(q=1,\cdots,\nu+1,\ t_{\nu+1}=0)$. A point

$$\chi_{j(1)}(t_0, t_1)\chi_{j(2)}(t_1, t_2)\cdots\chi_{j(\nu+1)}(t_\nu, 0)\rho$$

is called the end point (at $t = t_0$) of the trajectory. We say that the trajectory is admissible if two characteristic roots of adjacent bicharacteristic curves coincide at each joint. A bicharacteristic curve is also called a tracjectory of step 0 and it is always admissible.

For any integer $k \geqslant 0$ let $\Lambda^k(t_0\,;\rho)$ denote the set composed of the end points (at $t = t_0$) of admissible trajectories of step k, issuing from ρ. For any closed conic subset K of $T^*(R^n_x)$, we put

$$\Gamma^k(t_0\,;K) = \{\Lambda^j(t_0\,;\rho)\,;\rho\in K, j = 0, 1, \cdots, k\} \tag{5}$$

and

$$\widetilde{\Gamma}(t_0, K) = \bigcup_{k=0}^{\infty}\Gamma^k(t_0\,;K). \tag{6}$$

If we put $\rho_0 = (y, \eta)$ for a fixed $y \in R^n_x$ and $\eta = (1, 0, \cdots, 0)$ then all trajectories issuing from ρ_0 are admissible. In fact, if $\{(t, x(t), \xi(t))\,; t \in [0, T]\}$ denotes a bicharacteristic curve of λ_j $(j = 1, \cdots, \ell)$ through $(s, x, \xi) \in [0, T] \times T^*(R^n_x)$ satisfying $\xi = (\xi_1, 0, \cdots, 0)$ then $\xi'(t) = (\xi_2(t), \cdots, \xi_n(t))$ vanishes for any $t \in [0, T]$ and so we get $\lambda_j(t, x(t), \xi(t)) = a(t, x_1(t))\xi_1(t)$, $(j = 1, \cdots, \ell)$.

Theorem. *There exists an initial value G of* (C . P) *such that*

$$\mathrm{WF}\,G = \{(y, c\eta)\,; c > 0, \eta = (1, 0, \cdots, 0)\} \tag{7}$$

and for any $t_0 \in (0, T]$

$$\mathrm{WF}\,U(t_0) \supset \widetilde{\Gamma}(t_0\,;\mathrm{WF}\,G) \tag{8}$$

where $U(t)$ is the solution of (C . P) *for G.*

§ 2. **Proof of Theorem.** Let $h(t, x_1)$ be the solution of $D_t h + a(t, x_1)D_1 h = 0$, $h(0, x_1) = x_1$. Changing the valuables (t, x_1) to (t, h), we may assume $a = 0$. By means of the translation in R^n_x we may also assume $y = 0$. Let $G = {}^t(g, \cdots, g)$ and let

$$g(x) = \sum_{k=1}^{\infty} k^{-2}\phi_1(kx_1)\phi_2(kx')\exp ik^2x_1, \tag{9}$$

where $\phi_1, \phi_2 \in C_0^\infty$ satisfy

$$\phi_2(x'), \hat{\phi}_1(\xi_1), \hat{\phi}_2(\xi') \geqslant 0,\ \hat{\phi}_1(0) = \hat{\phi}_2(0) = 1.$$

As in Example 2.3 of [2] we have (7).

Let π denote the natural projection from $T^*(R_x^n)$ to $R_{x'}^{n-1}$. Since λ_j is linear in ξ and a_{jk} are independent of x_1, we can define a transformation $\pi_j(s,t) = \pi\chi_j(s,t)\pi^{-1}$ from R_x^{n-1} to R_x^{n-1}. Put $I_j(t,s)f(x) = f(x_1, \pi_j(s,t)x')$ for $f \in \mathscr{E}'$ and put

$$I_\phi(t,s) = \begin{bmatrix} I_1(t,s) & & 0 \\ & \ddots & \\ 0 & & I_\ell(t,s) \end{bmatrix}$$

Then, as in [4], [5] we have the fundamental solution for L of the form

$$E(t,s) = I_\phi(t,s) + \int_s^t I_\phi(t,\theta)W(\theta,s)d\theta,$$

$$W(t,s) = \sum_{\nu=1}^{\infty} W_\nu(t,s),$$

$$W_1(t,s) = B(t)I_\phi(t,s) = \left[b_{ij}(t)I_j(t,s); \begin{matrix} i \downarrow 1, \cdots, \ell \\ j \rightarrow 1, \cdots, \ell \end{matrix} \right]$$

$$W_{\nu+1}(t,s) = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} W_1(t,t_1)W_1(t_1,t_2)\cdots W_1(t_\nu,s)dt_\nu \cdots dt_2 dt_1, \quad \nu \geqslant 1.$$

Therefore, each component $u = u_r(t_0,x)$ $(r = 1, \cdots, \ell)$ of $U(t_0,x)$ is represented in the form

$$(10) \qquad u(t_0,x) = v(t_0,x) + \sum_{\nu=1}^{\infty} \sum_{J(\nu)\in\Pi_\nu} \int_0^{t_0} \int_0^{t_1} \cdots \int_0^{t_{\nu-1}} v_{J(\nu)}(t_0,t_1,\cdots,t_\nu;x)dt_\nu \cdots dt_1,$$

where $v = v_r(t_0,x) = g(x_1, \pi_r(0,t_0)x')$ and

$$v_{J(\nu)} = \beta_{j(1)}(t_0;x')\beta_{j(2)}(t_0,t_1;x')\cdots \beta_{j(\nu)}(t_0,\cdots,t_\nu;x')g(x_1, \pi_{J(\nu)}(t_0,\cdots,t_\nu)x').).$$

Here, $\pi_{J(\nu)}(t_0,\cdots,t_\nu) = \pi_{j(\nu+1)}(0,t_\nu)\pi_{j(\nu)}(t_\nu,t_{\nu-1})\cdots\pi_{j(1)}(t_1,t_0)$ and

$$\beta_{j(q)}(t_0,\cdots,t_{q-1};x') = b_{j(q)j(q+1)}(t_q, \pi_{j(q)}(t_q,t_{q-1})\cdots\pi_{j(1)}(t_1,t_0)x').$$

Let δ be the end point of the trajectory issuing from $\rho_0 = (0,\eta)$, corresponding to $J(\mu) \in \Pi_\mu^0$ and $\{t_1', \cdots, t_\mu'\}$, that is,

$$\delta = (0, \pi_{J(\mu)}{}^{-1}(t_0, t_1', \cdots, t_\mu')0, \eta).$$

We shall show $\delta \in \mathrm{WF}\, U(t_0)$. For any neighborhood $V = V_1 \times V_2$ of origin we take $\psi_1(x_1) \in C_0^\infty(V_1)$ and $\psi_2(x') \in C_0^\infty(V_2)$ such that $\psi_1, \psi_2 \geqslant 0$ and $\psi_1 = \psi_2 = 1$ in some neighborhoods of 0. Put

$$\psi_\mu(x) = \psi_1(x_1)\psi_2(\pi_{J(\mu)}(t_0, t_1', \cdots, t_\mu')x').$$

We check the value of Fourier transform of $\psi_\mu v_{J(\nu)}$ with respect to variables x at $(\xi_1, 0, \cdots, 0)$,

$$\begin{aligned}(11)\qquad &\hat{\psi_\mu v}_{J(\nu)}(t_0, t_1, \cdots, t_\nu; \xi_1, 0, \cdots, 0)\\ &= \sum_{k=1}^{\infty} k^{-2} w^k_{J(\nu)}(t_0, \cdots, t_\nu; \xi_1),\end{aligned}$$

where

$$\begin{aligned} w^k_{J(\nu)} &= \int (\exp i(k^2 - \xi_1)x_1)\psi_1(x_1)\phi_1(kx_1)dx_1\\ &\quad \times \int \beta\psi_2(\pi_{J(\mu)}x')\phi_2(k\pi_{J(\nu)}(t_0, t_1, \cdots, t_\nu)x')dx',\\ &\quad (\beta = \beta_{j(1)} \cdots \beta_{j(\nu+1)}).\end{aligned}$$

Since $\psi_1 = 1$ on $\mathrm{supp}\, \phi_1(kx_1)$ if k large enough, if follows from the hypotheses $\hat{\phi}_1, \phi_2, \psi_2$ and $b_{ij} \geqslant 0$ that

$$(12)\qquad w^k_{J(\nu)} \geqslant 0 \text{ for large } k.$$

Furthermore, if $(t_1, \cdots, t_\mu)$ is close to $(t_1', \cdots, t_\mu')$ and k large enough, then

$$(13)\qquad w^k_{J(\mu)}(t_0, t_1, \cdots, t_\mu; \xi_1) \geqslant C_\mu k^{-n} \hat{\phi}_1((\xi_1 - k^2)/k)\hat{\phi}_2(0)$$

for some constant $C_\mu > 0$ independent of k. Indeed, then we see that

$$\psi_2(\pi_{J(\mu)}(t_0, t_1', \cdots, t_\mu')x') = 1 \text{ on } \mathrm{supp}\, \phi_2(k\pi_{J(\mu)}(t_0, t_1, \cdots, t_\mu)x')$$

and so

$$(14)\qquad w^k_{J(\mu)}(t_0, t_1, \cdots, t_\mu, \xi_1) \geqslant k^{-1}\hat{\phi}_1((\xi_1 - k^2)/k) \int \phi_2(kx')\tilde{\beta}dx',$$

where

$$\tilde{\beta} = \beta(t_0, \cdots, t_\mu; \pi^{-1}_{J(\mu)}(t_0, \cdots, t_\mu)x') \left| \frac{\partial \pi_{J(\mu)}}{\partial x'}(x') \right|^{-1}.$$

Since it follows from (4) that for a constant $C_\mu > 0$ we have

$$\tilde{\beta} \geqslant C_\mu \text{ on } \mathrm{supp}\, \phi_2(kx')$$

if $(t_1, \cdots, t_\mu)$ is close to $(t_1', \cdots, t_\mu')$ and k large enough, we get (13) in view of (14).

Because the finite sum of the right hand side of (9) belongs to C_0^∞, we may only consider the sum, with respect to large k, of the right hand side of (11). Therefore, putting $\xi_1 = k_0^2$ for any large integer k_0, we obtain in view of (10)-(13)

$$\hat{\psi_\mu u}(t_0; k_0^2, 0, \cdots, 0) \geqslant C'_\mu k_0^{-2-n}$$

for some constant $C'_\mu > 0$ independent of k_0. This completes the proof.

§ 3. **Examples.** We mention examples in case $\ell = 2$.

Example 1. $\lambda_1 = \xi_1$, $\lambda_2 = \xi_1 + \xi_2$.

2. $\lambda_1 = \xi_1$, $\lambda_2 = \xi_1 + \xi_2 + t\xi_3$.

3. $\lambda_1 = \xi_1$, $\lambda_2 = \xi_1 + \xi_2 + t\xi_3 + t^2\xi_4$.

4. $\lambda_1 = \xi_1 + \xi_2$, $\lambda_2 = \xi_1 + \xi_3 + (t^2 - \alpha x_3 t)\xi_4$, constant $\alpha \leqslant 2$.

For these examples we shall clarify the set $\widetilde{\Gamma}(t_0; \mathrm{WF}G)$ for the initial value G of Theorem.

Example 1 belongs to the involutive characteristics case studied by Ludwig-Granoff [6] and many authors, and we have

$$\mathrm{WF}U(t_0) \subset \Gamma^1(t_0; \mathrm{WF}G) \tag{15}$$

for the solution $U(t)$ of (C. P) for any G. (See Theorem 3.4 of [5] or Theorem 6 of [8].) Since $\Gamma^1 \subset \widetilde{\Gamma}$ from the definition, for G of Theorem we have

$$\begin{aligned} \mathrm{WF}U(t_0) &= \Gamma^1(t_0; \mathrm{WF}G) \\ &= \{(y_1 + t_0, y_2 + s, y'', c\eta); c > 0, 0 \leqslant s \leqslant t_0\}, \end{aligned} \tag{16}$$

and incidently we have $\Gamma^1 = \widetilde{\Gamma}$, which directly follows from Frobenius' theorem. Ralston [9] showed that the statement similar to (16) when B satisfies $b_{12} = 1$ and others equal to 0. (See Example 1 of [9].) In his case we easily see that the representation (10) of the solution has only one integral term which does not vanish. It is clear from the proof in § 2 that we can weeken the hypothesis (4), for instance, $b_{ii} > 0$ can be replaced by $b_{ii} \geqslant 0$.

Ichinose [3] considered a class of hyperbolic systems with non-involutive characteristics which includes Example 2, and showed that for such systems

$$\mathrm{WF}U(t_0) \subset \widetilde{\Gamma}^2(t_0; \mathrm{WF}G),$$

where $\widetilde{\Gamma}^2$ is the right hand side of (5) with replaced $\Lambda^2(t_0;\rho)$ by the set composed of the end points of admissible trajectories of step 2 corresponding to only $J(2)=(1,2,1)$. Hence, we have for G of Theorem

$$\begin{aligned}\mathrm{WF}\,U(t_0) &= \widetilde{\Gamma}^2(t_0;\mathrm{WF}G)\\ &= \{(y_1+t_0, y_2+s_1-s_2, y_3+(s_1^2-s_2^2)/2, y'', c\eta);\\ &\qquad c>0, 0\leqslant s_2\leqslant s_1\leqslant t_0\}.\end{aligned}$$

This shows that $\mathrm{WF}\,U(t_0)$ is two dimension conic subset in $T^*(R^n_x)$.

As in Examples 1 and 2, in Examples 3 and 4 we have

$$\mathrm{WF}\,U(t_0)=\Gamma^3(t_0;\mathrm{WF}G)$$

for G of Theorem. This follows from the fact that for a class of hyperbolic systems containing Examples 3 and 4 we have

$$\mathrm{WF}\,U(t_0)\subset\Gamma^3(t_0;\mathrm{WF}G),$$

which will be proved in the forth coming paper by C. Iwasaki and the author.

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PROPAGATION PRECISE DES SINGULARITES POUR DES SYSTEMES HYPERBOLIQUES A CARACTERISTIQUES DE MULTIPLICITE VARIABLE

par

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INTRODUCTION

Considérons le système hyperbolique du type

$$(0.1)\qquad L = \begin{bmatrix} D_t + \lambda_1(t,x,D_x) & & 0 \\ & \ddots & \\ 0 & & D_t + \lambda_\ell(t,x,D_x) \end{bmatrix} + B(t,x) \quad ,$$

$$\text{sur } R_t^1 \times R_x^n \quad ,$$

où $\lambda_j(t,x,\xi)$ sont linéaires en ξ, c'est-à-dire,

$$(0.2)\qquad \lambda_j(t,x,\xi) = \sum_{k=1}^{n} a_{j,k}(t,x)\xi_k$$

$$(j = 1,\ldots,\ell) \ .$$

Ici $a_{j,k}(t,x) \in C^\infty$ sont à valeur réelle et les polynômes d'ordre 1 par rapport à x ;

$$(0.3)\qquad a_{j,k}(t,x) = \sum_{m=1}^{n} C_{j,k}^m(t)x_m + C_{j,k}^o(t) \ .$$

De plus, nous supposons que le terme d'ordre zéro $B(t,x) = (b_{j,k}(t,x)) \in C^\infty$ satisfait a la condition

(0.4) $$b_{j,k} \not\equiv 0 \quad \text{si} \quad j \neq k \ .$$

Nous considérons la propagation precise du front d'onde de la solution pour le problème de Cauchy

(P.C) $$L\,U = 0 \ , \quad U(0,x) = G \ , \quad G \in E' \ .$$

Dans [8] Kumano-go et Taniguchi ont donné une estimation de la propagation des singularités pour un système hyperbolique d'ordre 1 à partie principale diagonale (voir le Théorème 3.4 de [8]) en utilisant les opérateurs intégraux de Fourier à multi-phases. Leur estimation montre que le front d'onde de la solution peut se propager non seulement le long des courbes bicaractéristiques mais aussi le long de trajectoires admissibles, où on dit qu'une courbe obtenue en reliant des courbes bicaractéritiques est admissibles si deux racines caractéristiques des courbes bicaractéristiques adjacentes coïncident à chaque jointure (voir la Définition 1.1). Le Théorème 1 de cet article montre que le front d'onde d'une certaine solution se propage précisément le long des trajectoires admissibles sous des hypothèses quelconques. La démonstration du Théorème 1 est une généralisation de la méthode développée dans les §. 5-6 de [6], et donc le Théorème 2 de [6] est obtenu comme une application du Théorème 1 (voir aussi §.7 de [6]). Dans la dernière section , comme une autre application nous donnerons un exemple L sur $R_t \times R_x^1$ pour lequel il existe une donnée initiale G telle que

(0.5) $$\begin{cases} \text{supp sing } G = \{0\} \\ \text{supp sing } U(T) = [-1,1] \ , \end{cases}$$

où U est la solution de (P.C) et T est un positif fixé. Pour une équation hyperbolique d'ordre 2, Ichinose et Kumano-go ont montré dans [5] un résultat analogue à (0.5) en utilisant le résultat de Taniguchi-Tozaki [11], bien que l'on ne sache pas si le problème de Cauchy pour cette équation est bien posé jusqu'à t = T (voir Théorème 3.5 de [5]). Nous annonçons que dans l'étude à paraître la preuve du Théorème 1 sera appliquée à montrer la propagation précise des singularités dans le sens de la classe de Gevrey comme une remarque sur l'estimation donnée dans le Théorème 3.4 de [8].

1. Notations et résultat principal

On dit qu'une courbe

$$\left\{(t,x(t),\xi(t))\right\} \subset R^1_t \times T^*(R^n)$$

est la courbe bicaractéritique par rapport à $\lambda_j(t,x,\xi)$ passant par (s,y,η) si elle satisfait aux équations

$$(1.1) \qquad \begin{cases} dx/dt = d_\xi \lambda_j(t,x,\xi) \quad , \quad d\xi/dt = - d_x \lambda_j(t,x,\xi) \\ (x,\xi)|_{t=s} = (y,\eta) \ . \end{cases}$$

On désigne par $\chi_j(t,s)$ la transformation de l'espace cotangent $T^*(R^n)$ à lui-même telle que $(t,\chi_j(t,s)(y,\eta))$ soit la courbe bicaractéristique par rapport à λ_j

passant par (s,y,η). Puisque les λ_j sont linéaires en ξ et les polynômes d'ordre 1 par rapport à x, il existe une matrice $M_j(t,s)$ et un vecteur $d_j(t,s)$ tels que

$$\chi_j(t,s)(y,\eta) = (M_j(t,s)y + d_j(t,s),\ {}^tM_j(t,s)^{-1}\eta) \quad , \tag{1.2}$$

où ${}^tM_j^{-1}$ est l'inverse de la matrice transposée de $M_j(t,s)$.

Pour un entier $\nu \geqq 0$ on désigne par $J_\nu = (j_1,j_2,\ldots,j_q,\ldots,j_{\nu+1})$ une permutation répétée des éléments de $\{1,2,\ldots,\ell\}$, c'est-à-dire, $j_q = 1,2,\ldots,\ell$ pour $q = 1,\ldots,\nu+1$. Soit Π_ν l'ensemble de toute la J_ν. Soit Π_ν^o l'ensemble des J_ν vérifiant $j_q \neq j_{q+1}$, $1 \leqq q \leqq \nu$. Pour un $t_o > 0$ fixé on pose l'ensemble

$$\Delta_\nu^o = \{tt_\nu = (t_1,\ldots,t_\nu)\ ;\ t_o > t_1 > \ldots > t_\nu > 0\} \ , \ \nu \geqq 1 \ .$$

Une courbe continue $\{(t,x(t),\xi(t))\ ;\ t \in [0,t_o]\}$ s'appelle une trajectoire à l'étape $\nu \geqq 1$ partant de $\rho \in T^*(R^n)$ si pour une $J_\nu \in \Pi_\nu^o$ et un $tt_\nu \in \Delta_\nu^o$ elle est la courbe bicaractéristique par rapport à λ_{j_q} lorsque $t \in [t_q,t_{q-1}]$ et $(x(0),\xi(0)) = \rho$. On désigne par $C(J_\nu,tt_\nu,\rho)$ cette trajectoire partant de ρ pour une $J_\nu \in \Pi_\nu^o$ et un $tt_\nu \in \Delta_\nu^o$. Le point

$$\chi_{j_1}(t_o,t_1)\ \chi_{j_2}(t_1,t_2)\ \ldots\ \chi_{j_{\nu+1}}(t_\nu,0)\ \rho \tag{1.3}$$

s'appelle l'extrémité de la trajectoire $C(J_\nu,tt_\nu,\rho)$. Il s'ensuit de (1.2) que ce point est égal à

$$(M_{J_\nu}(tt_\nu)\,y + d_{J_\nu}(tt_\nu),\ {}^tM_{J_\nu}(tt_\nu)^{-1}\eta) \tag{1.4}$$

où une matrice $M_{J_\nu}(tt_\nu)$ et un vecteur $d_{J_\nu}(tt_\nu)$ sont définis par, respectivement,

$$M_{J_\nu}(tt_\nu) = M_{j_1}(t_o,t_1)\, M_{j_2}(t_1,t_2)\ldots M_{j_{\nu+1}}(t_\nu,0) \quad ,$$

$$d_{J_\nu}(tt_\nu) = d_{j_1}(t_o,t_1) + M_{j_1}(t_o,t_1)\, d_{j_2}(t_1,t_2)$$

$$+ \ldots + M_{j_1}(t_o,t_1)\ldots M_{j_\nu}(t_{\nu-1},t_\nu)\, d_{j_{\nu+1}}(t_\nu,0) \; .$$

Définition 1.1 : On dit que la trajectoire $C(J_\nu, tt_\nu, \rho)$ *est* ε*-admissible pour un petit* $\varepsilon \geqq 0$ *si l'on a*

$$|(\lambda_{j_q} - \lambda_{j_{q+1}})\,(t_q,x_q,\xi_q)| \leqq \varepsilon\,|\xi_q| \tag{1.5}$$

où $(x_q,\xi_q) = \chi_{j_{q+1}}(t_q,t_{q+1})\ldots\chi_{j_{\nu+1}}(t_\nu,0)\,\rho$. *On dit seulement admissible au lieu de 0-admissible.*

Remarquons qu'une courbe bicaractéristique est toujours une trajectoire admissible à l'étape 0. Pour $J_\nu = (j_1,\ldots,j_{\nu+1})$ et $J'_\mu = (j'_1,\ldots,j'_{\mu+1}) \in \Pi^o_\nu$ on écrit $J_\nu \supset J'_\mu$ s'il existe un sous-ensemble $\{q_1,q_2,\ldots,q_r,\ldots,q_{\mu+1}\ ;\ q_r < q_{r+1}\}$ de $\{1,\ldots,\nu+1\}$ tel que $j_{q_r} = j'_r$ pour $r = 1,\ldots,\mu+1$. Evidemment, si $J_\nu \supset J'_\mu$ alors $\nu \geqq \mu$.

Définition 1.2 : Pour un petit $\varepsilon_o > 0$ *et une trajectoire fixée* $C_\mu = C(J'_\mu, tt'_\mu, \rho)$ *on dit qu'une trajectoire* $C(J_\nu, tt_\nu, \rho)$ *est dans* ε_o*-voisinage de* C_μ *si* $J_\nu \supset J'_\mu$ *et les conditions suivantes sont vérifiées :*

Soient

$$J_\nu = (j_1,\ldots,j_q,\ldots,j_{\nu+1}) \ , \quad tt_\nu = (t_1,\ldots,t_q,\ldots,t_\nu) \ .$$

Au cas où $\mu \geq 1$, *soient*

$$J'_\mu = (j'_1,\ldots,j'_r,\ldots,j'_{\mu+1}) \ , \quad tt_\mu = (t'_1,\ldots,t'_r,\ldots,t'_\mu) \ .$$

Alors il existe un sous-ensemble $\{q_1,\ldots,q_\mu\}$ *de* $\{1,\ldots,\nu\}$ *tel que*

(1.6) $$|t_{q_r} - t'_r| < \varepsilon_o \qquad r = 1,\ldots,\mu \ .$$

Et de plus on obtient

(1.7) $$|t_q - t_{q-1}| < \varepsilon_o$$

pour tout q *satisfaisant aux*

(1.8) $$q_r \geq q > q - 1 \geq q_{r-1} \quad \text{et} \quad j_q \neq j'_r \ ,$$

où $r = 1,\ldots,\mu+1$, $q_o = 0$, $q_{\mu+1} = \nu+1$, $t_{\nu+1} = 0$.

Au cas où $\mu = 0$, *soit* $J'_\mu = (j)$. *Alors* $J_\nu = (j)$ *si* $\nu = 0$. *Si* $\nu \geq 1$ *on obtient* (1.7) *pour tout* $1 \leq q \leq \nu+1$ *satisfaisant à la condition* $j_q \neq j$.

Théorème 1 : Nous supposons :

(H-1) *Il existe une trajectoire admissible* $C_\mu = C(J'_\mu, tt'_\mu, \rho_o)$ *à l'étape* μ *partant de* $\rho_o = (y_o, \eta_o) \in T^*(R^n) \setminus 0$ *dont l'extrémité est* $\delta_o = (x_o, \xi_o)$ $(= (M_{J'_\mu}(tt'_\mu)\, y_o + d_{J'_\mu}(tt'_\mu),\ {}^t M_{J'_\mu}(tt_\mu)^{-1}\, \eta_o))$.

(H-2) *Si* $\mu \geq 1$ *alors on a*

$$\det\left(\partial_{t_p}\, \partial_{t_q}\, \xi_o \cdot (M_{J'_\mu}(tt'_\mu)\, y_o + d_{J'_\mu}(tt'_\mu))\right) \neq 0 \quad (p,q = 1,\dots,\mu) .$$

(H-3) *Soit* V_ε *un* ε*-voisinage ouvert conique de* ξ_o ,

$$V_\varepsilon = \{\xi \in R^n \setminus 0 \ ;\ |\xi/|\xi| - \xi_o/|\xi_o|| < \varepsilon\} .$$

Il existe un voisinage ouvert V_o *de* x_o *et un* $0 < \sigma \leq 1$ *tels que toute la trajectoire* ε*-admissible dont l'extrémité appartient à* $V_o \times V_\varepsilon$ *soit dans* ε^σ*-voisinage de* C_μ *si* $\varepsilon > 0$ *est suffisamment petit.*

Alors, il existe une donnée initiale G telle que

$$WF\, G = \{(y_o, c\,\eta_o) \ ;\ c > 0\} \tag{1.9}$$

et

$$WF\, U(t_o) \ni \delta_o \ , \tag{1.10}$$

où U est la solution de (P.C).

2. Démonstration du Théorème 1.

Il suffit de démontrer le Théorème au cas où le terme d'ordre zero B de L satisfait à la condition

$$(0.4)' \qquad b_{j,j} = 0 \quad , \quad b_{j,k} \not\equiv 0 \quad \text{si} \quad j \neq k \ .$$

En effet, soient $\alpha_j(t,x)$ les solutions des équations

$$(D_t + \lambda_j(t,x,D_x))\alpha_j + b_{jj}\alpha_j = 0 \quad , \quad \alpha_j(0,x) = 1 \ .$$

Si l'on pose une matrix

$$A(t,x) = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_\ell \end{bmatrix}$$

et pose $\tilde{U}(t,x) = A^{-1}U(t,x)$ pour la solution U de (P.C), on a le problème de Cauchy pour $\tilde{U}(t,x)$

$$(2.1) \qquad \begin{cases} L_0\tilde{U} + A^{-1}(L_0A + BA)\tilde{U} = 0 \\ \tilde{U}(0,x) = G \quad , \quad G \in E' \quad , \end{cases}$$

où L_0 est le terme principal de L. C'est facil de voir la matrix $A^{-1}(L_0A + BA)$ verifiant la condition comme (0.4)'. Puisque le multiplicateur A ne change pas de front d'onde de U, il suffit de considérer (2.1) au lieu de (P.C).

Soit π la projection naturelle de $T^*(R^n)$ à R^n. Si l'on pose une transformation de R^n à lui-même

$$(2.2) \qquad \pi_j(s,t) = \pi\, \chi_j(s,t)\pi^{-1} \quad ,$$

par (1.2) on obtient

(2.3) $$\pi_j(s,t)y = M_j(s,t)y + d_j(s,t)$$

$$= M_j(t,s)^{-1}(y - d_j(t,s)) \quad .$$

On pose $I_j(t,x)\ f(x) = f(\pi_j(s,t)x)$ pour $f \in E'$ et pose

$$I(t,s) = \begin{bmatrix} I_1(t,s) & & 0 \\ & \ddots & \\ 0 & & I_\ell(t,s) \end{bmatrix}$$

Alors, par la méthode itérative on obtient la solution fondamentale $E(t,s)$ pour L sous la forme

$$E(t,s) = I(t,s) + \int_s^t I(t,\theta)\ W(\theta,s)d\theta \quad ,$$

$$W(t,s) = \sum_{\nu=1}^{\infty} W_\nu(t,s) \quad , \quad W_1(t,s) = B(t)\ I(t,s) \quad ,$$

$$W_{\nu+1}(t,s) = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-1}} W_1(t,t_1)W_1(t_1,t_2) \cdots$$

$$W_1(t_\nu,s)\ dt_\nu \cdots dt_2 dt_1 \quad , \ \nu \geq 1 \ .$$

Donc, si l'on désigne par Δ_ν et $d\,\mathbb{t}_\nu$ la fermeture de Δ_ν^0 et $dt_1 \cdots dt_\nu$, respectivement, la solution $U(t_0,x) = {}^t(u_1(t_0,x),\ldots,u_\ell(t_0,x))$ de (P.C) est donnée sous la forme

(2.4) $$u_j(t_0,x) = g_j(\pi_j(0,t_0)x)$$

$$+ \sum_{\nu=1}^{\infty} \sum_{\substack{J_\nu \in \Pi_\nu \\ j_1 = j}} \int_{\Delta_\nu} \beta_{J_\nu}^{j}\, g_{j_{\nu+1}}(\pi_{J_\nu}(\mathbb{t}_\nu)x)d\mathbb{t}_\nu \quad (j = 1,\ldots,\ell) \quad ,$$

où

$$\pi_{J_\nu}(\tilde{t}_\nu) = \pi_{j_{\nu+1}}(0,t_\nu)\ \pi_{j_\nu}(t_\nu,t_{\nu-1}) \cdots \pi_{j_1}(t_1,t_0) \tag{2.5}$$

$$\beta^j_{J_\nu} \equiv \beta^j_{J_\nu}(\tilde{t}_\nu,x)$$

$$= \beta^j_{j_1}(t_0,x)\ \beta^j_{j_2}(t_0,t_1,x) \cdots \beta^j_{j_\nu}(t_0,\ldots,t_{\nu-1},x) \quad ,$$

et

$$\beta^j_{j_q}(t_0,\ldots,t_{q-1},x)$$

$$= -ib_{j_q,j_{q+1}}(t_q\ ,\ \pi_{j_q}(t_q,t_{q-1})\cdots\pi_{j_1}(t_1,t_0)x) \ , \quad i = \sqrt{-1} \ ,$$

$$(\ q = 1,\ldots,\nu\ ) \ .$$

Grace à (0.4)' on a

$$\begin{cases} \beta^j_{J_\nu} \not\equiv 0 & \text{si } J_\nu \in \Pi^0_\nu \\ \beta^j_{J_\nu} = 0 & \text{autrement .} \end{cases} \tag{2.6}$$

On choisit, comme donnée initiale de (P.C) ,

$$\begin{cases} G = {}^t(g_1,\ldots,g_\ell) \ , \ g_j = g \quad (\ j = 1,\ldots,\ell\ ) \\ g(x) = \sum_{k=1}^{\infty} n_k^{-2}\phi(n_k^3(x-y_0))\ \exp i\, n_k^5\,(x-y_0)\cdot\eta_0 \ , \end{cases} \tag{2.7}$$

où $\phi \in C_o^\infty$ telle que $\hat{\phi}(0) = 1$. On désigne par $\hat{\phi}$ la transformée de Fourier de ϕ. Ici les nombres n_k satisfont aux conditions

$$n_1 = 1 \quad , \quad n_{k+1} = 2c_o^2 \, n_k \quad (k = 1,2,\ldots) \quad , \tag{2.8}$$

où c_o est une constante positive telle que

$$c_o^{-1} < |M_{J_\nu}(tt_\nu)| < c_o \tag{2.9}$$

pour tout ν, $J_\nu \in \Pi_\nu^o$ et $tt_\nu \in \Delta_\nu^o$. La fonction g est une modification de celle donnée par Hörmander (voir Exemple 2.3 de [4]). Une autre modification a été utilisée dans [10].

On a :

$$WF\ g = \{(y_o, c\eta_o) \ ; \ c > 0\} \tag{2.10}$$

En effet, puisque chaque terme de g est dans C^∞ et les supports se contractent à y_o, on obtient

$$\text{supp sing } g = \{y_o\} \ . \tag{2.11}$$

Maintenant

$$\hat{g}(\xi) = \sum_{k=1}^{\infty} n_k^{-3n-2} \, \hat{\phi}\left((\xi - n_k^5 \, \eta_o) / n_k^3 \right) \exp -iy_0 \cdot \xi \tag{2.12}$$

Puisque

$$|n_{k_o}^5 - n_k^5| / n_k^3 \geq n_{k_o} \qquad \text{si } k \neq k_o \ , \tag{2.13}$$

pour tout $N > 0$ il existe une constante C_N telle que

$$|\hat{\phi}((n_{k_o}^5 - n_k^5)\eta_o / n_k^3)| \leqq C_N\, n_{k_o}^{-N} \qquad \text{si } k \neq k_o \quad .$$

Donc on obtient

$$|\hat{g}(n_{k_o}^5\, \eta_o)| \geqq n_{k_o}^{-3n-2}\, \hat{\phi}(0)/2 \qquad \text{si } k_o \text{ assez grand.} \tag{2.14}$$

Soit V un voisinage ouvert conique de η_o. Si $\xi \notin V$ il existe une constante C_V telle q

$$|\xi - n_k^5\, \eta_o| / n_k^3 \geqq C_V\, |\xi|^{2/5} \quad . \tag{2.15}$$

Alors, pour tout $N > 0$ il existe une constante C_N' telle que

$$|\hat{g}(\xi)| \leqslant C_N'\, |\xi|^{-N} \qquad \text{si } \xi \notin V \ . \tag{2.16}$$

Cette inégalité et (2.14) montrent (2.10).

Pour le voisinage V_o de x_o donné dans l'hypothèse (H-3) soit $\psi(x) \in C_o^\infty(V_o)$ telle que $\psi = 1$ près de x_o. Pour la démonstration il suffit de montrer que pour un j quelconque $(1 \leqq j \leqq \ell)$ on obtient

$$|\widehat{\psi u_j}(t_o, n_{k_o}^5\, \xi_o)| \geqslant c_1\, n_{k_o}^{-3n-2-(5/2)\mu} \ , \tag{2.17}$$

$$c_1 > 0 \ , \text{ si } k_o \text{ assez grand .}$$

Si l'on calcule la transformée de Fourier de ψu_j par (2.4) en tenant compte de (2.6), on a

(2.18) $$\widehat{\psi u}_j\,(t_o, n_{k_o}^5\,\xi_o) = \sum_k n_k^{-3n-2}\,(w_{(j)}(k) +$$

$$\sum_{\nu=1}^{\infty}\ \sum_{\substack{J_\nu \in \Pi_\nu^o \\ j_1 = j}} \int_{\Delta_\nu} w_{J_\nu}(k, tt_\nu)\,dtt_\nu)\ ,$$

où $w_{J_\nu} = G_{J_\nu}(k, tt_\nu) \exp i n_{k_o}^5\ F_{J_\nu}(tt_\nu) \qquad (\nu \geqq 0\ ,\ J_o = (j))\ ,$

(2.19) $$F_{J_\nu}(tt_\nu) = -\xi_o \cdot (M_{J_\nu}(tt_\nu)\, y_o + d_{J_\nu}(tt_\nu))\ ,$$

(2.20) $$G_{J_\nu}(k, tt_\nu)$$

$$= \int (\exp i y \cdot (n_k^5\,\eta_o - n_{k_o}^5\,{}^t M_{J_\nu}\,\xi_o) / n_k^3\ )$$

$$\psi(M_{J_\nu}(n_k^{-3}\,y + y_o) + d_{J_\nu})\ \phi(y)$$

$$\beta'_{J_\nu}\,dy$$

$$(\beta'_{J_\nu} = \beta^j_{J_\nu}\ |\det M_{J_\nu}| \neq 0)\ .$$

Ici on a utilisé le changement des variables x à

(2.21) $$\pi_{J_\nu}^{-1}\,(n_k^{-3}\,y + y_o) = M_{J_\nu}\,(n_k^{-3}\,y + y_o) + d_{J_\nu}\ .$$

Puisqu' il s'ensuit de (2.8) et (2.9) que

$$|n_{k_o}^5 \, {}^t M_{J_\nu} \xi_o - n_k^5 \eta_o| / n_k^3$$

$$= |(n_{k_o}^5 \, {}^t M_{J_\nu} \, {}^t M_{J'_\mu}^{-1} - n_k^5 \, I) \eta_o| / n_k^3 \geqq c_2 \, n_{k_o} \qquad (c_2 > 0) \text{ si } k \neq k_o \ ,$$

l'intégration par partie montre que pour tout $\nu \geqq 0$ et $N > 0$ il existe une constant C_N indépendante de ν telle que

$$|G_{J_\nu}(k, tt_\nu)| \leqq C_N^\nu \, n_{k_o}^{-N} \qquad \text{si } k \neq k_o \ . \tag{2.22}$$

Donc, pour montrer (2.17) il suffit de considérer le terme avec $k = k_o$ dans la somme de (2.18). Pour $J_\nu \in \Pi_\nu^o$ $(\nu \geqq 1)$ on pose

$$I_{J_\nu}(k_o) = \int_{\Delta_\nu} G_{J_\nu}(k_o, tt_\nu) \exp i n_{k_o}^5 \, F_{J_\nu}(tt_\nu) \, d\, tt_\nu \qquad (\nu \geqq 1) \ , \tag{2.23}$$

$$G_{J_\nu}(k_o, tt_\nu) \tag{2.20}'$$

$$= \int \exp i n_{k_o}^2 \, (\eta_o - {}^t M_{J_\nu} \, \xi_o) \cdot y$$

$$\psi(M_{J_\nu}(n_{k_o}^{-3} \, y + y_o) + d_{J_\nu}) \, \phi(y) \, \beta'_{J_\nu} \, dy \ .$$

On peut voir facilement que pour tout multi-indice α il existe une constante C_α telle que

$$|\partial_{tt_\nu}^\alpha \, G_{J_\nu}(k_o, tt_\nu)| \leqq C_\alpha^\nu \, n_{k_o}^{2|\alpha|} \ . \tag{2.24}$$

Aussi au cas où $\nu = 0$ on définit $I_{J_\nu}(k_o)$ par la fonction dans l'intégrale de (2.23). La démonstration sera achevée si l'on montre le

Lemme 2.1 : Soient $J_\nu \in \Pi_\nu^o$ *et* k_o *suffisamment grand. Si* $J_\nu \supset J'_\mu$ *alors il existe une constante* C_1 *telle que*

$$|I_{J_\nu}(k_o)| \leqslant C_1^\nu \, n_{k_o}^{-(5/2)\mu - \sigma(\nu-\mu)/2} / \nu! \quad . \tag{2.25}$$

Si $J_\nu \not\supset J'_\mu$ *, pour tout* $N > 0$ *il existe une constante* C_N *telle que*

$$|I_{J_\nu}(k_o)| \leqslant C_N^\nu \, n_{k_o}^{-N} / \nu \, ! \tag{2.26}$$

De plus il existe une constante $c_3 > 0$ *telle que*

$$|I_{J'_\mu}(k_o)| \geqq c_3 \, n_{k_o}^{-(5/2)\mu} \quad . \tag{2.27}$$

3. Preuve du Lemme 2.1.

Pour la simplicité on écrit k au lieu d'écrire k_o et aussi n_{k_o} dans cette section. Selon cette convention on considère

$$I_{J_\nu}(k) = \int_{\Delta_\nu} G_{J_\nu}(k, tt_\nu) \exp i k^5 F_{J_\nu}(tt_\nu) \, d\, tt_\nu \quad , \tag{3.1}$$

$$J_\nu \in \Pi_\nu^o \quad , \quad \nu \geqq 0 \quad ,$$

où la notation de l'intégrale est omise si $\nu = 0$, $F_{J_\nu}(tt_\nu)$ a été définie par (2.19) et

$$G_{J_\nu}(k, tt_\nu) = \int (\exp i k^2 (\eta_o - {}^t M_{J_\nu} \xi_o) \cdot y) \tag{3.2}$$

$$\psi(M_{J_\nu}(k^{-3} y + y_o) + d_{J_\nu}) \, \phi(y) \, \beta'_{J_\nu} \, dy \, .$$

Pour tout α il existe une constante C_α indépendante de k telle que

$$|\partial_{tt_\nu}^{\alpha} G_{J_\nu}| \leqslant C_\alpha^\nu k^{2|\alpha|} \quad . \tag{3.3}$$

Soit k suffisamment grand ci-dessous. En considération des supports de $\phi(y)$ et $\psi(M_{J_\nu}(k^{-3} y + y_o) + d_{J_\nu})$ dans le terme à droite de (3.2) on voit que

$$M_{J_\nu}(tt_\nu) y_o + d_{J_\nu}(tt_\nu) \in V_o \qquad (\nu \geqq 1), \tag{3.4}$$

$$\text{si} \quad tt_\nu \in \text{supp } G_{J_\nu} \cap \Delta_\nu$$

et

$$M_j(t_o,0)y_o + d_j(t_o,0) \in V_o \qquad \text{si } G_{(j)} \neq 0 \quad . \tag{3.4}'$$

D'abord on va considérer le cas où $\mu \geqq 1$. Puisque la courbe bicaractéristique est admissible et que l'on a $(j) \neq J_\mu$, , il s'ensuit de (3.4)' et de l'hypothèse (H-3) que

$$\xi_o \neq {}^t M_j(t_o,0)^{-1} \eta_o \quad \text{si } G_{(j)} \neq 0 \quad . \tag{3.5}$$

Donc, au moyen de l'intégration par partie par rapport à y on voit que pour tout $N > 0$ il existe une C_N telle que

$$|G_{(j)}| \leqq C_N k^{-2N} \quad , \tag{3.6}$$

ce qui montre l'inégalité (2.26) avec $\nu = 0$.

Soit κ un petit paramètre choisi par la suite. Soit $h(s) \in C^\infty$ telle que $0 \leq h \leq 1$, $h = 1$ pour $s \leq 1$ et $h = 0$ pour $s \geq 2$. Si l'on pose

$$\psi_{J_\nu}(tt_\nu) = h(\kappa^{-3\nu} k|\eta_0 - {}^t M_{J_\nu}(tt_\nu)\xi_0|) \quad , \quad \nu \geq 1 \, , \tag{3.7}$$

l'intégration par partie par rapport à y montre que

$$|G_{J_\nu}(k, tt_\nu)| \leq C_N^\nu k^{-N} \tag{3.8}$$

$$\text{si} \quad tt_\nu \in \text{supp}\,(1 - \psi_{J_\nu}) \quad ,$$

parce que l'on a

$$|\eta_0 - {}^t M_{J_\nu}(tt_\nu)\xi_0| \geq \kappa^{3\nu}/k$$

$$\text{sur} \quad \text{supp}\,(1 - \psi_{J_\nu}) \quad .$$

Puisque le volume de Δ_ν est $t_0^\nu/\nu!$ et (3.8) est vérifié, pour la preuve il suffit de considérer $G'_{J_\nu} = \psi_{J_\nu} G_{J_\nu}$ au lieu de G_{J_ν} dans (3.1). Pour G'_{J_ν} on obtient l'inégalité comme (3.3).

Remarquons que si $tt_\nu \in \text{supp}\, G'_{J_\nu} \cap \Delta_\nu$ alors l'extrémité de $C(J_\nu, tt_\nu, \rho_0)$ appartient à $V_0 \times V_\varepsilon$ avec $\varepsilon = \kappa^\nu/k$. En effet, c'est la conséquence de (3.4) et

$$|{}^t M_{J_\nu}^{-1} \eta_0 - \xi_0| \leq \kappa^{2\nu}/k \quad \text{sur supp}\, \psi_{J_\nu} \, , \tag{3.9}$$

où on a utilisé l'inégalité $2\, c_0 \kappa^\nu << 1$ pour κ assez petit.

Soit $E(J_\nu)$ un sous-ensemble de Δ_ν défini par

$$E(J_\nu) = \{ tt_\nu \in \text{supp } G'_{J_\nu} \cap \Delta_\nu \ ; \ C(J_\nu , tt_\nu , \rho_0)$$

$$\text{est } \kappa^\nu / k \text{ - admissible} \} \ .$$

Il s'ensuit de (H-3) que si $J_\nu \not\supset J'_\mu$ alors $E(J_\nu) = \emptyset$. Par ailleurs, au cas où $J_\nu \supset J'_\mu$, si $tt_\nu \in E(J_\nu)$ on obtient (1.6) et (1.7) avec $\varepsilon_0 = (\kappa^\nu / k)^\sigma$. Remarquons que le nombre des possibilités du sous-ensemble $\{q_1,\ldots,q_\mu\}$ $(= Q_\mu)$ satisfaisant à (1.6) est inférieur à 2^ν. On désigne par Q^j_μ $(j=1,\ldots,n_\nu \ , \ n_\nu \leqq 2^\nu)$ tous les sous-ensembles. Pour un sous-ensemble $Q_\mu = \{q_1,\ldots,q_\mu\}$ fixé, soient q'_j $(j=1,\ldots,n'_\nu)$ tous les éléments de $\{1,\ldots,\nu\}$ satisfaisant (1.8). Evidemment, le nombre n'_ν est supérieur à $[(\nu - \mu + 1) / 2]$. On pose

$$(3.10) \qquad \varphi_{J_\nu}(tt_\nu) = \prod_{r=1}^{\mu} h(|t_{q_r} - t'_r| / \varepsilon_0) \prod_{j=1}^{n'_\nu} h(|t_{q'_j} - t_{q'_j - 1}| / \varepsilon_0)$$

où $\varepsilon_0 = (\kappa^\nu / k)^\sigma$. Pour chaque Q^j_μ si l'on définit $\varphi^j_{J_\nu}$ comme φ_{J_ν} , il s'ensuit de (H-3) que

$$(3.11) \qquad E(J_\nu) \cap \left(\bigcap_{j=1}^{n_\nu} \text{supp } (1 - \varphi^j_{J_\nu}) \right) = \emptyset \ .$$

Pour le moment nous supposons $J_\nu \supset J'_\mu$. Soient

$$(3.12) \qquad \begin{cases} G^1_{J_\nu} = G'_{J_\nu} \varphi^1_{J_\nu} \\ G^j_{J_\nu} = G'_{J_\nu} \varphi^j_{J_\nu} \prod_{m=1}^{j-1} (1 - \varphi^m_{J_\nu}) \ , \ 2 \leqq j \leqq n_\nu \\ G''_{J_\nu} = G'_{J_\nu} \prod_{m=1}^{n_\nu} (1 - \varphi^m_{J_\nu}) \ , \end{cases}$$

pour lesquelles on obtient les inégalités comme (3.3). Alors on a

$$I'_{J_\nu}(k) = \sum_{j=1}^{n_\nu} I^j_{J_\nu}(k) + I''_{J_\nu}(k) \quad , \tag{3.13}$$

où I'_{J_ν} , $I^j_{J_\nu}$, I''_{J_ν} sont définis par (3.1) avec G_{J_ν} remplacée par G'_{J_ν} , $G^j_{J_\nu}$, G''_{J_ν} , respectivement. Il s'ensuit de (3.11) que si $tt_\nu \in \operatorname{supp} G''_{J_\nu}$ alors il existe un q quelconque ($1 \leqq q \leqq \nu$) tel que

$$|\partial_{t_q} F_{J_\nu}(tt_\nu)| \geqq \kappa^\nu / 2k \quad . \tag{3.14}$$

Pour montrer cette inégalité on appelle une proposition qui est la conséquence immédiate du Théorème 2.3 de Kumano-go - Taniguchi - Tozaki [7].

Proposition 3.1 : Soient $\phi_j(t,s) = \phi_j(t,s\,;x,\xi)$ les solutions des équations eiconales

$$\partial_t \phi_j + \lambda_j(t,x,d_x\phi_j) = 0 \quad , \quad \phi_j(t,s)\big|_{t=s} = x\cdot\xi \quad (j=1,\dots,\ell) \quad . \tag{3.15}$$

Pour une suite des phase-fonctions

$$\phi_{j_1}(t_0,t_1) \ , \ \phi_{j_2}(t_1,t_2),\dots,\phi_{j_{\nu+1}}(t_\nu,0)$$

($J_\nu = (j_1,\dots,j_{\nu+1})$, $tt_\nu \in \Delta_\nu$) si l'on désigne par $\Phi_{J_\nu}(t_0,tt_\nu,0\,;x,\xi)$ le multi-produit

$$(\phi_{j_1}(t_0,t_1) \# \phi_{j_2}(t_1,t_2) \# \dots \# \phi_{j_{\nu+1}}(t_\nu,0))\,(x,\xi) \quad ,$$

alors on obtient

(3.16) $$\partial_{t_q} \Phi_{J_\nu}(t_o, \tt_\nu, 0; x, \xi) \Big|_{\substack{x = z^\nu \\ \xi = \eta}}$$

$$= (\lambda_{j_q} - \lambda_{j_{q+1}})(t_q, x_q, \xi_q) \qquad q = 1, \ldots, \nu \ ,$$

<u>où</u> $(x_q, \xi_q) = \chi_{j_{q+1}}(t_q, t_{q+1}) \ldots \chi_{j_{\nu+1}}(t_\nu, 0)(y, \eta)$ <u>et</u>

$$z^\nu = \pi \chi_{j_1}(t_o, t_1) \ldots \chi_{j_{\nu+1}}(t_\nu, 0)(y, \eta) \ .$$

Puisque $\phi_j(t,s\,;x,\xi) = (\pi_j(s,t)x)\cdot\xi = M_j^{-1}(t,s)\,(x - d_j(t,s))\cdot\xi$ on obtient

$$\Phi_{J_\nu}(t_o, \tt_\nu, 0\,; x, \xi) = (\pi_{J_\nu}(\tt_\nu)\, x)\cdot\xi$$

$$= M_{J_\nu}(\tt_\nu)^{-1}\ (x - d_{J_\nu}(\tt_\nu))\cdot\xi \ .$$

Alors on obtient

(3.17) $$\partial_{t_q} \Phi_{J_\nu} \Big|\ x = d_{J_\nu} + M_{J_\nu} y_o \ , \ \xi = \eta_o$$

$$= (\xi_o - {}^t M_{J_\nu}^{-1} \eta_o) \cdot \partial_{t_q} (d_{J_\nu} + M_{J_\nu} y_o) + \partial_{t_q} F_{J_\nu} \ .$$

Donc, en considération de (3.16) et (3.9) on obtient (3.14) en notant que le premier terme à droite de (3.17) est inférieur à $\kappa^\nu / 2k$ si l'on choisit κ suffisamment petit.

On va montrer que pour tout $N > 0$ il existe une C_N telle que

(3.18) $$|I''_{J_\nu}(k)| \leq C_N^\nu\, k^{-N} / \nu ! \ .$$

Pour un k fixé, soit $\sum_j \psi^j_{J_\nu}(\tau_\nu) \equiv 1$ la partition de l'unité sur supp $G''_{J_\nu} \cap \Delta_\nu$ telle que le nombre de $\psi^j_{J_\nu}$ dont les supports se superposent mutuellement soit inférieur à 2^ν et on a

$$|\tau_\nu - \tau'_\nu| \leq \kappa^{2\nu}/k \quad \text{si} \quad \tau_\nu, \tau'_\nu \in \text{supp}\ \psi^j_{J_\nu} \ , \tag{3.19}$$

$$|\partial^\alpha_{\tau_\nu} \psi^j_{J_\nu}| \leq C^\nu_\alpha k^{|\alpha|} \ , \tag{3.20}$$

pour une constante C_α indépendante de ν et k. En effet, on peut faire une telle partition de l'unité en posant

$$\psi^j_{J_\nu} = h_j / \sum_j h_j \ ,$$

$$h_j(\tau_\nu) = h(|\tau_\nu - \tau^j_\nu| \kappa^{-2\nu} k)$$

pour des points $\tau^j_\nu \in$ supp $G''_{J_\nu} \cap \Delta_\nu$ convenables. Par (3.14) et (3.19) on peut voir que pour tout $\psi^j_{J_\nu}$ il existe un q $(1 \leq q \leq \nu)$ tel que

$$|\partial_{t_q} F_{J_\nu}| \geq \kappa^\nu / 3k \quad \text{sur} \quad \text{supp}\ \psi^j_{J_\nu} \ , \tag{3.21}$$

parce que $\kappa^{2\nu} << \kappa^\nu$. Remarquons que

$$\int_{\Delta_\nu} . \, d\tau_\nu = \int_{\Delta_{\nu-1}} dt_1 \ldots dt_{q-1}\, dt_{q+1} \ldots dt_\nu \int_{t_{q+1}}^{t_{q-1}} dt_q \ . \tag{3.22}$$

Au moyen de (3.21) il s'ensuit de l'intégration par partie que

$$(3.23)\qquad \int_{t_{q+1}}^{t_{q-1}} e^{ik^5 F_\nu}\, \psi_\nu^j\, G_\nu''\, dt_q$$

$$= \Big[-ik^{-5}\, e^{ik^5 F_\nu}\, \psi_\nu^j\, G_\nu''\, (\partial_{t_q} F_\nu)^{-1} \Big]_{t_q = t_{q+1}}^{t_q = t_{q-1}}$$

$$+ ik^{-5} \int_{t_{q+1}}^{t_{q-1}} e^{ik^5 F_\nu}\, \partial_{t_q} (\psi_\nu^j\, G_\nu''\, (\partial_{t_q} F_\nu)^{-1}) dt_q \quad ,$$

où pour la simplicité on a écrit ν au lieu de J_ν . En notant que

$$(3.24)\qquad F_{J_\nu}(tt_\nu)\Big|_{t_q = t_{q+1}} = F_{J^q_{\nu-1}}(t_1,\ldots,t_{q-1}, t_{q+1},\ldots,t_\nu)$$

pour $J^q_{\nu-1} = (j_1,\ldots,j_{q-1}, j_{q+1},\ldots,j_{\nu+1})$, par (3.21)-(3.23) on obtient

$$I''_{J_\nu}(k) = k^{-4} \sum_q \tilde{I}_{J^q_{\nu-1}}(k) + k^{-2}\, \tilde{I}_{J_\nu}(k) \quad ,$$

où le premier terme à droite est la somme par rapport à tous les q satisfaisants à (3.21) pour chaque $\psi^j_{J_\nu}$. Ici $\tilde{I}_{J^q_{\nu-1}}(k)$, $\tilde{I}_{J_\nu}(k)$ sont définis, comme dans (3.1), avec les phases $F_{J^q_{\nu-1}}$, F_{J_ν} et les amplitudes $\tilde{G}_{J^q_{\nu-1}}$, $\tilde{G}_{J_\nu}$ satisfaisant aux inégalités comme (3.3), respectivement, parce que le nombre de la superposition des $\psi^j_{J_\nu}$ est inférieur à 2^ν et on a (3.20) et (3.21). Si $j_{q-1} \neq j_{q+1}$ dans $J^q_{\nu-1}$, on peut utiliser le procédé ci-dessus pour $\tilde{I}_{J^q_{\nu-1}}(k)$ au lieu de $I''_{J_\nu}(k)$ parce que l'on a la propriété

comme (3.14) si $tt_{\nu-1} \in \text{supp } \tilde{G}^{q}_{J_{\nu-1}} \cap \Delta_{\nu-1}$. Remarquons que pour un $J_\nu \in \Pi_\nu$ on obtient

$$F_{J_\nu}(tt_\nu) = F_{J'_{\nu-1}}(tt'_{\nu-1}) \quad \text{si} \quad j_q = j_{q+1} \quad , \tag{3.25}$$

où $J'_{\nu-1}$ et $tt'_{\nu-1}$ sont définis en omettant j_q et t_q dans J_ν et tt_ν, respectivement. Donc, si l'on définit (3.1) aussi pour $J_\nu \in \Pi_\nu / \Pi^o_\nu$ alors on a

$$I_{J_\nu}(k) = I_{J'_{\nu-1}}(k) \quad , \quad \text{si} \quad j_q = j_{q+1} \tag{3.26}$$

où l'amplitude de $I_{J'_{\nu-1}}(k)$ est donnée sur la forme

$$\int_{t_{q+1}}^{t_{q-1}} G_{J_\nu}(k, tt_\nu)\, d\, t_q \quad .$$

Il s'ensuit de la propriété (3.26) avec $J_\nu = J^q_{\nu-1}$ que l'on peut utiliser la méthode pour $I''_{J_\nu}(k)$ aussi au cas où $j_{q-1} = j_{q+1}$ dans $J^q_{\nu-1}$. En notant que (2.26) avec $\nu = 0$ est vérifiée, on a (3.18) si l'on répète le procédé ci-dessus pour $\tilde{I}_{J_\nu}(k)$.

Au cas où $J_\nu \neq J'_\mu$, le procédé qui donne (3.18) est applicable à montrer (2.26) pour $I_{J_\nu}(k)$, parce que $E(J_\nu) = \emptyset$ et donc on a (3.14) si $tt_\nu \in \text{supp } G'_{J_\nu}$.

Pour obtenir (2.25) il suffit de montrer que

$$|I^j_{J_\nu}(k)| \leq C^\nu\, k^{-(5/2)\mu - \sigma(\nu-\mu)/2} / \nu \,! \tag{3.27}$$

pour une constante C indépendante de k et ν, parce qu'on a $n_\nu \leqq 2^\nu$. Nous considérons un $I^j_{J_\nu}(k)$. Pour la simplicité on écrit $j = 0$. En considération de (H-3) et le support de φ_{J_ν} définie par (3.10) on peut voir qu'il existe deux sous-ensembles $\{p_1, \ldots, p_\mu\}$

et $\{p'_1,\ldots,p'_\mu\}$ de $\{1,\ldots,\nu\}$ tels que

$$(3.28)\qquad \begin{cases} p_j \leqq q_j \leqq p'_j & (j = 1,\ldots,\mu)\ , \\ p'_j < p_{j+1} & (j = 1,\ldots,\mu-1) \end{cases}$$

et avec $\varepsilon_0 = (\kappa^\nu / k)^\sigma$ on a

$$(3.29)\qquad |t_{p_r} - t_{p'_r}| \leqq \nu\,\varepsilon_0\ ,\quad r = 1,\ldots,\mu$$

$$(3.30)\qquad \begin{cases} |t_{p_r} - t_{p_r-1}| \geqq \varepsilon_0 \\ |t_{p'_r} - t_{p'_r+1}| \geqq \varepsilon_0\ ,\quad r = 1,\ldots,\mu,\ t_{\nu+1} = 0 \end{cases}$$

$$\text{si } tt_\nu \in \operatorname{supp} G^0_{J_\nu} \cap \Delta_\nu\ .$$

Remarquons que le nombre des possibilités des sous-ensembles $\{p_1,\ldots,p_\mu\}$ $(= P_\mu)$ et $\{p'_1,\ldots,p'_\mu\}$ $(= P'_\mu)$ est inférieur à 4^ν. On désigne par $(P^j_\mu\ ,\ P'^j_\mu)$ $(j = 1,\ldots,n''_\nu\ ,\ n''_\nu \leqq 4^\nu)$ tous les couples de deux sous-ensembles.

Pour un $(P_\mu\ ,\ P'_\mu)$ on pose

$$(3.31)\qquad f_{J_\nu}(tt_\nu) = \prod_{r=1}^{\mu} (1-h)\,(|t_{p_r} - t_{p_r-1}| / \varepsilon_0) \times$$

$$(1-h)\,(|t_{p'_r} - t_{p'_r+1}| / \varepsilon_0)\ .$$

Si l'on définit $f^j_{J_\nu}$ pour chaque $(P^j_\mu\ ,\ P'^j_\mu)$ comme f_{J_ν}, alors on a par (H-3)

$$\text{supp } G^{o}_{J_\nu} \cap \Delta_\nu \cap \left(\bigcap_{j=1}^{n''_\nu} \text{supp } (1 - f^{j}_{J_\nu}) \right) = \emptyset$$

Donc, on obtient

$$I^{o}_{J_\nu}(k) = \sum_{j=1}^{n''_\nu} I^{oj}_{J_\nu}(k) \quad , \tag{3.32}$$

où $I^{oj}_{J_\nu}$ sont définies par (3.1) avec G_{J_ν} remplacée par

$$G^{o,1}_{J_\nu} = G^{o}_{J_\nu} f^{1}_{J_\nu} \quad , \quad G^{oj}_{J_\nu} = G^{o}_{J_\nu} f^{j}_{J_\nu} \prod_{m=1}^{j-1} (1 - f^{m}_{J_\nu}) \qquad \nu \geqq 2 \quad ,$$

pour lesquelles on a les inégalités comme (3.3).

Nous considérons un $I^{oj}_{J_\nu}(k)$. Pour la simplicité on écrit $j = 0$. On prend le changement de variable $tt_\nu = (t_1,\ldots,t_\nu)$ à $(S_\mu, W_{\nu_1}, V_{\nu_2})$, ci-dessous : on pose

$$S_\mu = (s_1,\ldots,s_\mu)$$

$$s_r = (t_{p_r} + t_{p_{r+1}} + \ldots + t_{p'_r}) / (p'_r - p_r + 1) \quad ,$$

et pour tous les couples $(q,q-1)$ $(q = 1,\ldots,\nu)$ sauf $(p_r, p_r - 1)$ $(r = 1,\ldots,\nu)$ on pose

$$w_q = t_{q-1} - t_q$$

si $|t_{q-1} - t_q| \leqslant \nu\, \varepsilon_o$ sur $\text{supp } G^{o,o}_{J_\nu}$ et pose $v_q = t_{q-1} - t_q$ autrement. On pose $W_{\nu_1} = (w_1,\ldots,w_{\nu_1})$ et $V_{\nu_2} = (v_1,\ldots,v_{\nu_2})$. En considération de (3.10) on obtient

$$\nu_1 \geqq n'_\nu \geqq [(\nu - \mu + 1) / 2] \quad . \tag{3.33}$$

Par le changement des variables ci-dessus on a

$$I_{J_\nu}^{o,o}(k) = \int_{D_1} dW_{\nu_1} \int_{D_2} dV_{\nu_2} \int \tilde{G}_{J_\nu}^{o,o} \exp i k^5 \tilde{F}_{J_\nu} \, dS_\mu \tag{3.34}$$

où $\tilde{G}_{J_\nu}^{o,o}(k,S_\mu,W_{\nu_1},V_{\nu_2}) = G_{J_\nu}^{o,o}(k,tt_\nu)$, $\tilde{F}_{J_\nu}(S_\mu,W_{\nu_1},V_{\nu_2}) = F_{J_\nu}(tt_\nu)$,

$$D_1 = \left\{ w_j \geqq 0 \; ; \; \sum_{j=1}^{\nu_1} w_j \leqq \nu^2 (\kappa^\nu / k)^\sigma \right\} \quad \text{et}$$

$$D_2 = \left\{ v_j \geqq 0 \; ; \; \sum_{j=1}^{\nu_2} v_j \leqq t_o \right\}.$$

Au moyen de (3.24) et (3.25), il s'ensuit de (H-3) que

$$\tilde{F}_{J_\nu}(S_\mu, 0, V_{\nu_2}) = F_{J'_\mu}(S_\mu) \, . \tag{3.35}$$

En considération de (3.16) et (3.17) on peut voir par (H-2) et (H-3) que

$$S_\mu = tt'_\mu \iff \partial_{s_q} F_{J'_\mu}(S_\mu) = 0 \, , \; q = 1,\ldots,\mu \, , \tag{3.36}$$

si $S_\mu \in \pi^*$ supp $\tilde{G}_{J_\nu}^{o,o}$, où π^* est la projection naturelle de $R_S \times R_W \times R_V$ à R_S . Par (H-2) on obtient

$$\det [\partial_{s_p} \partial_{s_q} \tilde{F}_{J_\nu}(S_\mu, W, V)] \neq 0 \tag{3.37}$$

$$\text{sur supp } \tilde{G}_{J_\nu}^{o,o} \cap R_S \times D_1 \times D_2$$

si l'on choisit κ suffisamment petit. Grâce à (3.36) et (3.37), le théorème de la fonction implicite montre l'existence de la solution unique $S_\mu(W_{\nu_1},V_{\nu_2})$ de

$$\partial_{s_q} \tilde{F}_{J_\nu}(\cdot, W_{\nu_1}, V_{\nu_2}) = 0 \ , \quad q = 1, \ldots, \mu$$

si $(W_{\nu_1}, V_{\nu_2}) \in D_1 \times D_2$. En regardant (W_{ν_1}, V_{ν_2}) comme paramètre, la méthode de la phase stationnaire (voir §.1-2 de [2]) donne

$$I_{J_\nu}(k, W_{\nu_1}, V_{\nu_2}) = \int \tilde{G}^{o,o}_{J_\nu} \exp i k^5 \tilde{F}_{J_\nu} \, dS_\mu \tag{3.38}$$

$$\sim (2\pi / k^5)^{\mu/2} \, |\det Q|^{-1/2} \, e^{(\pi i/4)\operatorname{sgn} Q}$$

$$\times \sum_{j=0}^{\infty} (R^j \tilde{G}^{o,o}_{J_\nu}) \, k^{-5j} / j! \ , \quad k \to \infty \ ,$$

où $Q \equiv Q(W_{\nu_1}, V_{\nu_2}) = [\partial_{s_p} \partial_{s_q} \tilde{F}_{J_\nu}(S_\mu(W_{\nu_1}, V_{\nu_2}), W_{\nu_1}, V_{\nu_2})]$, $R = i < Q^{-1} \partial_{S'_\mu}, \partial_{S'_\mu} > / 2$ et $\tilde{G}^{o,o}_{J_\nu}(S'_\mu, W_{\nu_1}, V_{\nu_2}) = \tilde{G}^{o,o}_{J_\nu}(S_\mu, W_{\nu_1}, V_{\nu_2})$. Ici $(S_\mu, W_{\nu_1}, V_{\nu_2}) \to S'_\mu$ est une C^∞-application telle que

$$S'_\mu(S_\mu, W_{\nu_1}, V_{\nu_2}) = S_\mu - S_\mu(W_{\nu_1}, V_{\nu_2}) + O(|S_\mu - S_\mu(W_{\nu_1}, V_{\nu_2})|^2) \ ,$$

$$\tilde{F}_{J_\nu}(S_\mu, W_{\nu_1}, V_{\nu_2}) = \tilde{F}_{J_\nu}(S_\mu(W_{\nu_1}, V_{\nu_2}), W_{\nu_1}, V_{\nu_2}) + < Q \, S'_\mu \, , \, S'_\mu > / 2 \ .$$

Puisque les volumes de D_1 et D_2 sont

$$(\nu^2 (\kappa^\nu / k)^\sigma)^{\nu_1} / \nu_1! \leq k^{-\sigma\nu_1} / \nu_1 \quad \text{et} \quad t_o^{\nu_2} / \nu_2! \ ,$$

respectivement, par (3.38) et (3.33) on peut voir que $I^{0,0}_{J_\nu}(k)$ est inférieur au terme à droite de (3.27). Donc, par (3.32) on a (3.27), ce qui montre (2.25).

Puisque n_ν de (3.13) et n''_ν de (3.32) sont 1 au cas où $J_\nu = J'_\mu$, (3.38) donne (2.27) si l'on montre qu'il existe une constante $c_4 > 0$ telle que

$$|G_{J'_\mu}(k, \tilde{t}'_\mu)| \geqq c_4 \quad ; \tag{3.39}$$

En considération de (2.6) et le changement des variables (2.21) on voit que $\beta'_{J'_\mu}$ est presque égale à une non-zéro constante sur le support de $\phi(y)$ lorsque k suffisamment grand. Donc, en notant que $\xi_0 = {}^t M_{J'_\mu}(\tilde{t}'_\mu)^{-1} \eta_0$, par (3.2) (ou (2.20)') on obtient (3.39) si k assez grand. La démonstration au cas où $\mu \geqq 1$ est achevée.

Au cas où $\mu = 0$, si $(j) = J'_\mu$, c'est-à-dire, la trajectoire C_μ est la courbe bicaractéristique par rapport à λ_j, il existe une constante $c_5 > 0$ telle que

$$|G_{(j)}(k)| \geqq c_5 \quad \text{si k assez grand} \quad , \tag{3.39}'$$

ce qui montre (2.27). L'inégalité (2.26) est obtenue de la même manière que le cas o $\mu \geqq 1$. Si l'on change la définition de $\varphi_{J_\nu}(\tilde{t}_\nu)$ à

$$\varphi_{J_\nu}(\tilde{t}_\nu) = \prod_{j=1}^{n'_\nu} h(|t_{q'_j} - t_{q'_{j-1}}| / \varepsilon_0) \quad \text{avec } \varepsilon_0 = (\kappa^\nu / k)^\sigma \quad , \tag{3.10}'$$

le deuxième terme à droite de (3.13) est estimé par le terme à droite de (2.26) et le premier terme à droite de (3.13) est estimé par le terme à droite de (2.25) parce que le volume de $\Delta_\nu \cap \operatorname{supp} \varphi_{J_\nu}$ est inférieur à $t_0^\nu k^{-\sigma\nu/2} / \nu!$. Donc on obtient (2.25) aussi au cas où $\mu = 0$.

Remarque 1. Il s'ensuit du Lemme 2.1 que si U est la solution de (P.C) par la donnée initiale G définie par (2.7) on obtient

$$(3.40) \qquad c_6 \leqq |\widehat{\psi U}(t_o, k^5 \xi_o)|\ k^{3n+2+(5/2)\mu} \leqq c_6^{-1}$$

$$\text{si } k \geqq k_1 \text{ , } k_1 \text{ assez grand ,}$$

où la fonction $\psi \in C_o^\infty(V)$ a été donnée au-dessus de (2.17) et c_6 est une constante positive indépendante de k. Ici, pour la simplicité, on a écrit k au lieu d'écrire n_{k_o}. En vertu de la formule (3.38) et la partie dernière de la preuve on voit

$$(3.41) \qquad c_6 = c_7^\mu\ |\det(\partial_{t_p}\ \partial_{t_q}\ \xi_o \cdot M_{J'_\mu}(tt'_\mu)\, y_o + d_{J'_\mu}(tt'_\mu))|^{-1/2},\ (p,q = 1,\ldots,\mu) \quad ,$$

où c_7 est une constante positive indépendante de μ et le choix de $\psi \in C_o^\infty(V)$ telle que $\psi = 1$ près de x_o. Remarquons que le nombre k_1 dépend du choix de ψ.

Remarque 2. La démonstration du Théorème 1 est valable aussi au cas où il existe plusieurs trajectoires admissibles qui relient ρ_o avec $\delta_o = (x_o, \xi_o)$ si les nombres de leurs étapes sont différents mutuellement, la condition (H-2) est vérifiée pour chaque trajectoire admissible et la condition suivante est vérifiée ;

(H-3)' | Soit V_ε un ε-voisinage ouvert conique de ξ_o. Il existe un voisinage ouvert V_o de x_o et un $0 < \sigma \leqq 1$ tels que toute trajectoire ε-admissible dont l'extrémité appartient à $V_o \times V_\varepsilon$ soit dans ε^σ-voisinage d'une trajectoire admissible quelconque donnée ci-dessus si $\varepsilon > 0$ est suffisamment petit.

En effet, c'est clair si l'on considère que la contribution par rapport à la

trajectoire asmissible à l'étape le plus petit est dominante.

4. Application du Théorème 1.

On considère un exemple de L avec $\ell = 2$ sur $R^1_t \times R^1_x$ dont λ_1 et λ_2 sont définies par

$$\lambda_1 = \alpha(t)\xi \ , \quad \lambda_2 = -\lambda_1 \ , \tag{4.1}$$

où $\alpha(t)$ est une C^∞ fonction telle que

$$\begin{cases} \alpha(\beta_j) = 0 \ , \quad (-1)^j \alpha(t) > 0 \quad \text{si } \beta_j < t < \beta_{j+1} \ , \\ \alpha'(\beta_j) \neq 0 \ , \quad \displaystyle\int_{\beta_j}^{\beta_{j+1}} |\alpha(t)|dt = 2^{-(j+1)} \ , \end{cases} \tag{4.2}$$

pour une suite $\{\beta_j\}$ $(j = 0,1,2,\ldots)$ satisfaisant aux

$$0 = \beta_0 < \beta_1 < \ldots < \beta_j < \ldots < T$$

$$\beta_j \to T \qquad (j \to \infty) \ .$$

Ici T est un positif fixé. On va montrer l'existence de cette fonction α. Soient

$$a_j = 2^{-(j+1)/\log(j+3)} \ , \quad b_j = a_j^{\log(j+3)-1} \ ,$$

$$(j = 0,1,2,\ldots) \ .$$

Alors on a $\displaystyle\sum_{j=0}^{\infty} a_j < +\infty$,

(4.3) $$a_j < a_{j-1} < 2\,a_j \quad , \quad b_j < b_{j-1} < 2\,b_j \quad .$$

Soit $f(t)$ une C^∞ fonction telle que

$$\begin{cases} f(0) = -2 \quad , \quad f(2) = 1 \quad , \quad f'(1) = 1 \quad , \\ f \neq 0 \quad \text{dans } (0,2) \text{ sauf } 1 \\ \int_0^1 f\,dt = -1/2 \quad , \quad \int_1^2 f\,dt = 1/4 \quad , \\ f^{(n)}(0) = f^{(n)}(2) = 0 \quad \text{pour tout entier } n \geq 1 \quad . \end{cases}$$

Soit $h(t)$ une C^∞ fonction telle que

$$\begin{cases} h(0) = 0 \quad , \quad h(1) = 1 \quad , \quad h \neq 0 \quad \text{pour } t \geq 0 \\ h^{(n)}(0) = h^{(n)}(1) = 0 \quad \text{pour tout entier } n \\ \int_0^1 h\,dt = 1 \quad . \end{cases}$$

Si l'on pose pour $j = 1,2,\ldots$

$$\varphi_j(t) = \begin{cases} (-1)^{j-1}\, b_{j-1}\, f((t+a_{j-1})/a_{j-1}) \quad \text{sur } [0,a_{j-1}] \\ (-1)^{j}((b_{j-1}-2b_j)\, h(2(t-a_{j-1})/a_j) - b_{j-1}) \text{ sur } (a_{j-1}\,,\, a_{j-1}+a_j/2) \\ (-1)^{j}\, b_j\, f((t-a_{j-1}-a_j/2)a_j) \quad \text{sur } [a_{j-1}+a_j/2,\, a_{j-1}+3a_j/2] \quad , \end{cases}$$

alors on obtient $\int_0^{\tau_j} \varphi_j\,dt = (-1)^{j-1}\,2^{-j}$, $j = 1,2,\ldots$, où $\tau_j = a_{j-1}+3a_j/2$, parce

qu'il s'ensuit de (4.3) que $b_{j-1} - 2b_j < 0$. Soient $\beta_0 = 0$,

$$\beta_j = \sum_{k=1}^{j} \tau_k \quad (j = 1,2,\ldots) \ , \quad T = \sum_{k=1}^{\infty} \tau_k \ .$$

Alors on obtient une fonction $\alpha \in C^\infty$ désirée sous la forme

$$\alpha(t) = \varphi_j(t - \beta_{j-1}) \quad \text{si } \beta_{j-1} \leqq t < \beta_j \ ,$$

(4.4)

$$\alpha(t) = 0 \quad \text{si } t < 0 \text{ ou } t > T \ ,$$

parce que $\varphi_{j+1}^{(n)}(0) = \varphi_j^{(n)}(\tau_j)$ et $\lim_{j \to \infty} \max_t |\varphi_j^{(n)}| = 0$ pour tout entier $n \geqq 0$. On remarque qu'une fonction analogue à $\alpha(t)$ a été donnée indépendamment par [5].

Pour cet exemple (4.1) si l'on considère $t_0 = \beta_j$ $(j = 1,2,\ldots)$ on a 2^j trajectoires admissibles partant de $(0,\eta_0)$, $\eta_0 \neq 0$ dont les extrémités $(\pm (2k-1)2^{-j}, \eta_0)$, $k = 1,\ldots,2^{j-1}$. Toute trajectoire admissible satisfait aux hypothèses du Théorème 1. En effet, (H-1) et (H-2) sont vérifiées parce que chaque trajectoire admissible est isolée dans le sens de la Définition 1.2. Par $\alpha'(\beta_k) \neq 0$ $(k < j)$ on voit que (H-2) est vérifiée. En vertu du Théorème 1 on voit qu'il existe une donnée initiale G telle que

$$WF\, G = \{(0, c\eta_0) \ ; \ c > 0\}$$

(4.5)

$$WF\, U(\beta_j) = \{(\pm (2k-1)2^{-j} \ , \ c\eta_0) \ ;$$

$$k = 1,\ldots,2^{j-1} \ , \ c > 0\} \ ,$$

où U est la solution de (P.C). Ici on a utilisé le Théorème 3.4 de [8] pour montrer

que le terme à gauche de (4.5) contient celui à droite. Puisque WF U est l'ensemble fermé dans $T^*(R_t \times R_x)$, on obtient

Théorème 2 : Soit L le système dans cette section. Alors il existe une donnée initiale G telle que

$$(4.6) \qquad \begin{cases} WF\, G = \{(0, c\eta_0) \ ; \ c > 0\} \\ WF\, U(T) = [-1,1] \times \{c\, \eta_0\} \end{cases}$$

où U est la solution de (P.C).

Corollaire : Pour G et U dans le Théorème 2 on obtient

$$(4.7) \qquad \begin{cases} supp\ sing\ G = \{0\} \ , \\ supp\ sing\ U(T) = [-1,1] \ . \end{cases}$$

Comme il est dit dans l'introduction, le résultat analogue à (4.6) a été donné par [5] en utilisant l'ouvrage de [11], c'est-à-dire, la propagation précise des singularités pour l'équation

$$(4.8) \qquad \partial_t^2 - t^{2\ell}\, \partial_x^2 - a\, t^{\ell-1}\, \partial_x \ ,$$

où a est une constante et $\ell = 1,2,\ldots,$ (cf. [1],[3]).

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PROPAGATION DU FRONT D'ONDE GEVREY POUR UN EXEMPLE DE SYSTEME HYPERBOLIQUE

par

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INTRODUCTION

Dans [5] Kumano-go et Taniguchi ont donné une description de la propagation du front d'onde C^∞ de la solution du problème de Cauchy hyperbolique

$$\text{(P.C)} \qquad L\,U = 0\ ,\ U(0,x) = G\ ,\ G \in H^{-\infty} \equiv \bigcup_{s=-\infty}^{\infty} H^s\ ,$$

où L est un système hyperbolique d'ordre 1 à partie principale diagonale (voir Théorème 3.4 de [5] ou Théorème 0 de [3]). Leur description

$$\text{(0.1)} \qquad WF\,U(t_0) \subset \Gamma(t_0;\ WF\ G)\ ,\ t_0 > 0$$

semble assez forte en apparance parce que $\Gamma(t_0\ ;\ WF\ G)$ contient un point d'accumlation de l'ensemble composé des extremités de toutes les trajectoires ε-admissibles partant d'un ε-voisinage conique de WF G pour $0 < \varepsilon < 1$. Nous référons à [3] et [4] pour la définition de la trajectoire ε-admissible et la terminologie employée dans cet article. Remarquons que le point d'accumulation n'est pas en général contenu dans la fermeture de l'ensemble des extremités de toutes les trajectoires 0-admissibles partant de WF G, c'est à dire,

$$\text{(0.2)} \qquad \begin{cases} \textit{il existe un } \delta_\infty \in T^*(R^n) \setminus 0 \textit{ et un sous-ensemble} \\ \textit{compact conique } \tilde{K} \textit{ de } T^*(R^n) \textit{ tels que} \\ \\ \delta_\infty \in \Gamma(t_0\ ;\ \tilde{K}) \setminus \overline{\left(\bigcup_{k=1}^{\infty} \Gamma^k(t_0\ ;\ \tilde{K})\right)} \end{cases}$$

Dans [3] Iwasaki et l'auteur ont donné un exemple de système L ayant la propriété (C.2) (voir la fin de § 7 de [3]).

Cet article est consacré à montrer que la description (0.1) est optimale pour un autre exemple L (voir (2.1) dans § 2) ayant la propriété (0.2), si l'on considère la propagation du front d'onde dans le sens d'une certaine classe de Gevrey au lieu de considérer celle du front d'onde C^∞ . Plus précisément, pour l'exemple défini par (2.1) vérifiant la propriété (0.2) on va montrer qu'il existe une donnée initiale G de (P.C) telle que pour tout $1 < \theta < 3$ on obtient

$$\text{(0.3)} \qquad WF_\theta G = \tilde{K} \ , \quad WF_\theta U(t_0) \ni \delta_\infty \ ,$$

(voir Théorème 2.2 dans § 2) ,

où l'on a désigné par $WF_\theta u$ le front d'onde θ-Gevrey de $u \in \mathcal{D}'$ défini par Hörmander [1] (voir la Définition 1.1 dans § 1). Nous remarquons que la description (0.1) est vérifiée même si l'on remplace le front d'onde C^∞ par celui θ-Gevrey pour $\theta \geq 2$ en supposant que le symbole de l'opérateur pseudo-différentiel L soit analitique. Ce résultat sera démontré dans l'étude à paraître.
On ne sait pas encore si la description (0.1) est vérifiée avec le front d'onde θ-Gevrey pour $1 \leq \theta < 2$ à la place du front d'onde C^∞.

1. Front d'onde θ-Gevrey.

Soit Ω un ouvert dans R^n et soit $\theta \geq 1$, alors on désigne par $G^\theta(\Omega)$ l'ensemble de toute fonction Gevrey d'ordre θ , i.e. $u \in C^\infty(\Omega)$ et pour tout compact K de Ω il existe une constante C telle que pour tout multi-indice α

$$\text{(1.1)} \qquad \sup_{x \in K} |D^\alpha u(x)| \leq C^{|\alpha|+1}(\alpha!)^\theta \ .$$

Selon la Définition 3.1 de Hörmander [1] nous définissons le front d'onde θ-Gevrey ($\theta \geq 1$).

Définition 1.1. *Soit* $(x_0, \xi_0) \in T(R^n) \setminus 0$ *et soit* $u \in D'$. *On dit que* (x_0, ξ_0) *est dans le complémentaire du front d'onde* θ*-Gevrey* ($\theta \geq 1$) *de* u *si et seulement s'il existe un voisinage ouvert* U_0 *de* x_0 , *un voisinage ouvert conique* Γ_0 *de* ξ_0 *et une suite bornée* $\{u_N\} \in E'$ *telle que*

$$(1.2) \qquad \begin{cases} u_N = u \quad \text{dans} \quad U_0 \\ |\widehat{u_N(\xi)}| \leq C^{N+1}(N!)^{\theta}|\xi|^{-N} , \quad N = 0,1,\ldots, \end{cases}$$

pour une constante $C > 0$ *indépendante de* N . *On désigne par* $WF\ u$ *le front d'onde* θ*-Gevrey de* u .

Remarque. Au cas où $\theta = 1$, la définition ci-dessus est celle du front d'onde analytique.

De la même façon dans la preuve du Corollaire 1.4 du Chapteur V de Treves [6] on obtient le lemme suivant en tenant compte (2.3) de [1] .

Lemme 1.2. *Si et seulement si* $(x_0, \xi_0) \in T^*(R^n) \setminus 0$ *n'appartient pas au front d'onde* θ*-Gevrey de* $u \in D'$, *il existe un voisinage ouvert* U_0 *de* x_0 *et un voisinage ouvert conique* Γ_0 *de* ξ_0 *telle que l'on ait la propriété suivante: Pour toute suite des fonctions* $\{h_N\}$ ($N = 1,2,\ldots$) *dans* $C_0^{\infty}(U_0)$ *telle que*

$$(1.3) \qquad |D^{\alpha}h_N| \leq (C_1 N)^{\theta|\alpha|} , \quad |\alpha| \leq N$$

pour une constante $C_1 > 0$ *indépendante de* N , *on obtient*

(1.4) $$|\widehat{(h_N u)}(\xi)| \leq C_2^{N+1}(N!)^\theta |\xi|^{-N} \quad , \quad \xi \in \Gamma_0$$

pour une constante C_2 *indépendante de* N .

Dés lors, on obtient

Corollaire. *Soient* $\theta > 1$, $(x_0, \xi_0) \in T(R^n) \setminus 0$ *et* $u \in D'$. *Supposons qu'il existe une suite croissantes des constantes* $\{C_N\}$, *une suite* $\{\zeta_N ; \zeta_N \in R^n \setminus 0\}$ *et une suite des fonctions* $\{h_N\}$ *dans* C^∞ *vérifiant (1.3)* ($N = 1,2,\ldots$) *telles que*

(1.5) $$\begin{cases} C_N \to \infty \quad (N \to \infty) \ , \\ \zeta_N/|\zeta_N| \to \xi_0/|\xi_0| \quad , \ |\zeta_N| \to \infty \quad (N \to \infty) \ , \end{cases}$$

(1.6) $$\text{supp } h_k \to \{x_0\} \qquad (N \to \infty)$$

et

(1.7) $$|\widehat{(h_N u)}(\zeta_N)| \geq C_N^{N+1}(N!)^\theta |\zeta_N|^{-N} \ , \ , \quad N = 1,2,\ldots .$$

Alors (x_0, ξ_0) *appartient à* $WF_\theta\, u$.

Remarque. Soit $\{N_k\}$ une sous-suite de $\{1,2,\ldots\}$. Evidemment, on obtient la même conclusion même si l'on remplace $\{N\}$ par $\{N_k\}$ dans le Corollaire.

2. Exemple maniaque de système hyperbolique et résultat.

Soit L un systeme hyperbolique de la forme

$$(2.1) \qquad L = \begin{bmatrix} D_t + \lambda_1(t,x,D_x) & 0 \\ 0 & D_t + \lambda_2(t,x,D_x) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ,$$

$$\text{sur } R_t \times R_x^4 ,$$

où

$$(2.2) \qquad \begin{cases} \lambda_1(t,x,\xi) = 0 \\ \lambda_2(t,x,\xi) = \xi_1 + x_1\xi_2 + x_2\xi_3 + (t^4 - 8x_1t^3 + 48x_2t^2 - 192x_3t)\xi_4 . \end{cases}$$

Pour un entier pair $\mu \geq 2$ et un positif fixé $t_0 > 0$ on pose

$$(2.3) \qquad \rho_\mu = (0,\eta_\mu) \in T^*(R^4) \quad , \quad \eta_\mu = (-(t_0/2\mu)^4, 0, 0, 1) .$$

Alors on a

Proposition 2.1. *Des trajectoires admissible, partant de* ρ_μ *, sont seulement* $\mu+2$ *trajectiores, c'est à dire, deux courbes bicaractéritiques par rapport à* λ_1 *,* λ_2 *, et des trajectoires* $C(J'_r , \not{t}'_r , \rho_\mu)$ *à l'étape* $r = 1,\ldots,\mu$. *Ici* $J'_R = (1,2,1,\ldots,1)$ *ou* $(2,1,2,\ldots,1)$ *selon* r *pair ou impair, et* $\not{t}'_r = (t_1^r,\ldots,t_r^r) = (\sigma_r,\ldots,\sigma_1)$, $\sigma_q = (2q-1)\beta$ ($q = 1,\ldots,\mu$, $\beta = (t_0/2\mu)$). *De plus, pour la trajectoire* $C_\mu = C(J'_\mu , \not{t}'_\mu , \rho_\mu)$ *toutes les hypothèse du Théorème 1 de* [4] *sont vérifiées avec* ρ_μ *à la place de* ρ_0 *et avec* $\sigma < 1$ *dans l'hypothèse (H-3). (Voir Fig. 1).*

Admetton cette proposition; elle sera démontrée dans §4.

On désigne par δ_μ l'extremité de la trajectoire $C(J'_\mu , \mathfrak{t}'_\mu , \rho_\mu)$. Considérons une suite des extremités $\{\delta_\mu\}$ $(\mu = 2, 4, \dots)$.

Théorème 2.2. Soit δ_∞ un point d'accumulation de $\{\delta_\mu\}$. Si l'on pose

$$\tilde{K} = \{(0, c\eta_0) \in T^*(R^4) \setminus 0 \ ; \ c > 0 \ , \ \eta_0 = (0,0,0,1)\} \ , \tag{2.4}$$

on obtient

$$\delta_\infty \in \Gamma(t_0 \ ; \ \tilde{K}) \setminus \overline{\left(\bigcup_{k=1}^{\infty} \Gamma^k(t_0 \ ; \ \tilde{K}) \right)} \tag{2.5}$$

De plus il existe une donnée initiale G telle que pour tout $1 < \theta < 3$ on ait

$$WF_\theta \, G = \tilde{K} \tag{2.6}$$

$$WF_\theta \, U(t_0) \ni \delta_\infty \ , \tag{2.7}$$

où U est la solution de (P.C) pour L défini par (2.1) et (2.2) .

3. Plan de la démonstration du Théorème 2.2.

Prenons une sous-suite $\{\delta_{\mu_k}\}$ de $\{\delta_\mu\}$ telle que $\delta_{\mu_k} \to \delta_\infty$ et posons $\eta^k = \eta_{\mu_k}$, c'est à dire,

$$\eta^k = (-(t_0/2\mu_k)^4, 0, 0, 1) \ . \tag{3.1}$$

Posons

$$\phi(z) = (2\pi)^{-2} \exp - \sum_{j=1}^{4} z_j^2/2 \quad , \quad z \in \mathbb{C}^4$$

et

$$g(z) = \sum_{k=1}^{\infty} n_k^{-2} \, \phi(n_k^\gamma z) \exp i n_k^5 \, z \cdot \eta^k \quad , \tag{3.2}$$

où γ est une constante $3 \geq \gamma > 5/2$ et $\{n_k\}$ est une suite criossante qui

qui sera exigée quelques conditions par la suite. Premièrement, on demande

$$(3.3) \qquad n_1 = 1 \ , \quad n_{k+1} > 2c_o^2 n_k \quad (k = 1,2,\dots) \ ,$$

où c_o est la même constante dans (2.9) de [4].

On choisit, comme donnée initiale G de (P.C),

$$(3.4) \qquad G(x) = {}^t(g(x),g(x)) \ ,$$

où $g(x)$ est la restriction de $g(z)$ à R^4 et cette fonction est une modification de l'Exemple 2.3 de Hörmander [2]. Alors, de la presque même façon dans [4], on a

$$(3.5) \qquad WF_{\theta_o} g = \{(0,c\eta_o) \ ; \ c > 0 \ , \ \eta_o = (0,0,0,1) \}$$

où $\theta_o = (2 - 2\gamma/5)^{-1} > 1$. En effet, remarquons que $\phi(z) \exp -iz\cdot\eta$ est bornée dans un ouvert de $\mathbf{C}^4$

$$\Omega_o = \{ z \in \mathbf{C}^4 \ ; \ |\mathrm{Re}\ z|^2 > |\mathrm{Im}\ z|^2 + 2|\mathrm{Im}\ z||\eta| \} \ .$$

Puisqu'il s'ensuit de $\Sigma\, n_k^{-2} < +\infty$ que la somme à driote de (3.2) convèrge uniformément dans Ω_o , la fonction $g(z)$ est analytique dans Ω_o . Donc, $g(x)$ est analytique réelle hors de l'origin. Maintenant, on a

$$\hat{g}(\xi) = \sum_{k=1}^{\infty} n_k^{-14} \hat{\phi}((\xi - n_k^5 \eta^k)/ n_k^{\gamma}) \ .$$

Puisque

$$|n_{k_o} - n_k| > 1 \quad \text{si} \quad k \neq k_o$$

et que

$$|n_{k_o}^5 - n_k^5| / n_k^{\gamma} = |n_{k_o} - n_k| \times \sum_{r=0}^{4} n_k^{r-\gamma} n_{k_o}^{4-r}$$

$$> n_k^{3-\gamma} n_{k_o} \geq n_{k_o} \quad , \quad \text{si} \quad k \neq k_o \ ,$$

on obtient

$$|\hat{\phi}((n_{k_o}^5 \eta^{k_o} - n_k^5 \eta^k) / n_k^\gamma)| \leq \exp -n_{k_o}^2/2 \ , \quad \text{si} \quad k \neq k_o$$

où on a utilisé le fait élémentaire

$$\hat{\phi}(\xi) = \exp -|\xi|^2/2 \quad . \tag{3.6}$$

Donc, on a

$$|\hat{g}(n_{k_o}^5 \eta^{k_o})| \geq n_{k_o}^{-14}/2 \quad , \text{ si } k_o \text{ assez grand ,}$$

ce qui montre

$$WF_{\theta_o} g \supset WF\, g \supset \{(0, c\eta_o) \ ; \ c > 0\} \ .$$

Soit V un voisinage ouvert conique de η_o. Si $\xi \notin V$, il existe deux constantes C_V et $C_V' > 0$ telles que

$$|\xi - n_k^5 \eta^k| / n_k^\gamma \geq C_V(|\xi| + n_k^5 \eta^{k_o})/n_k^\gamma$$

$$\geq C_V' |\xi|^{(5-\gamma)/5} \ .$$

Donc , par (3.6) on obtient

$$|\hat{g}(\xi)| \leq \exp -c_o|\xi|^{2-2\gamma/5} \tag{3.7}$$

pour une constante $c_o > 0$, ce qui montre l'autre direction d'inclusion de (3.5).

Poson $\delta_{\mu_k} = (x^k, \xi^k)$ et $\delta_\infty = (x_o, \xi_o)$. Soit $\{f_N(x)\}$ ($N = 1,2,\ldots$) une suite des fonctions dans C_o^∞ telle que $0 \leq f_N \leq 1$, $f_N = 1$ sur $|x - x_o| \leq 1$, $f_N = 0$ hors de $|x - x_o| \geq 2$ et pour tout α

$$|D^\alpha f_N| \leq (CN)^{|\alpha|} \quad , \quad |\alpha| \leq N \tag{3.8}$$

pour une constante $C > 0$ indépendante de N . Il est élémentaire de construire f_N (voir Lemme 1.1 du Chapteur V de [6]). Soit $\{\gamma_N\}$ une suite criossante des nombres telle que $\gamma_N < N^{\theta-1}$. Si l'on pose

$$h_N(x) = f_N(x_o + \gamma_N(x - x_o)) ,$$

la suite $\{h_N(x)\}$ vérifie les propriétés (1.3) et (1.6). Poson $N_k = \mu_k/2$. On choisit une sous-suite $\{\gamma_{N_k}\}$ doucement criossante telle que

$$h_{N_k}(x^k) = 1 \quad . \tag{3.9}$$

On montra que pour tout k il existe une constante $c > 0$ indépendante de k telle que pour la solution $U = {}^t(u_1, u_2)$ on ait

$$|(\widehat{h_N u_1})(t_o,\zeta_k)| \geqq c^{N_k+1} N_k^{3N_k} |\zeta_k|^{-N_k-14/5} , \tag{3.10}$$

où $\zeta_k = n_k^5 \xi^k$. En admettant ce résultat, par le Corollaire dans §1 et sa Remarque on peut obtenir la dérnière partie du Théorème 2.2. La première du Théorème 2.2 sera démontrée dans §5 avec la Proposition 2.1.

4. Preuve de (3.10).

Pour la simplicité, posons $h_k = h_{N_k}$. De la même façon que celle pour obtenir (2.18) de [4],

$$(\widehat{h_{k_o} u_1})(t_o , n_{k_o}^5 \xi^{k_o}) = \sum_k^\infty n_k^{-14} \left(w_1(k) + \right. \tag{4.1}$$

$$\left. \sum_{\nu=1}^{\infty} \sum_{\substack{J_\nu \in \Pi_\nu^o \\ j_1=1}} \int_{\Delta_\nu} w_{J_\nu}(k, \tau_\nu)\, d\tau_\nu \right) ,$$

où $w_{J_\nu} = G_{J_\nu}(k, \mathrm{tt}_\nu) \exp i n_{k_o}^{5} F_{J_\nu}(\mathrm{tt}_\nu)$ $(\nu \geq 0 , J_o = (1))$. Ici

(4.2) $$F_{J_\nu} = -\xi^{k_o} \cdot d_{J_\nu}(\mathrm{tt}_\nu) \quad (\text{notons que } y_o = 0)$$

et

(4.3) $$G_{J_\nu}(k, \mathrm{tt}_\nu)$$

$$= \int_{\Delta_\nu} \exp iy \cdot (n_k^{5} \eta^k - n_{k_o}^{5}\, {}^t M_{J_\nu} \xi^k)/n_k^{\gamma})$$

$$h_{k_o}(M_{J_\nu}(n_k^{-\gamma} y) + d_{J_\nu}) \phi(y) \beta'_{J_\nu} dy .$$

On obtient $|\beta'_{J_\nu}| = 1$ parce que $|\beta^j_{J_\nu}| = 1$ et que $\det M_{J_\nu} = 1$ découle du calcul direct. De la presque même manière que celle pour obtenir (2.22) de [4], pour tout $\nu \geq 0$ et tout k on a

(4.4) $$|G_{J_\nu}(k, \mathrm{tt}_\nu)| \leq C_{k_o}^{\nu} n_{k_o}^{-5N_{k_o}-15} \quad , \text{ si } k \neq k_o$$

avec une constante C_{k_o} indépendante de ν et k. Notons que par (3.3) on obtient $\Sigma\, n_k^{-14} < +\infty$ et que le volume Δ_ν est majoré par $t_o^\nu/\nu!$. Dès lors, si l'on désigne par $R(k_o)$ la somme par rapport à $k \neq k_o$ du terme à droite de (4.1), il existe une constante C'_{k_o} telle que

(4.5) $$|R(k_o)| \leq C'_{k_o} n_{k_o}^{-5N_{k_o}-15} .$$

On considère le terme avec $k = k_o$ à droite de (4.1). Puisqu'il s'ensuit de la Proposition 2.1 que pour la trajectiore $C(J'_\mu , \mathrm{tt}'_\mu , \rho_\mu)$ avec $\mu = \mu_{k_o}$ toutes les hypothèses sont vérifiées, on peut utiliser la méthode pareille à celle dans §3 de [4] et obtenir le Lemme 2.1 de [4]. Remarquons que l'on a

propriété (2.24) de [4] avec $n_{k_o}^{(5-\gamma)|\alpha|}$ à la place de $n_{k_o}^{2|\alpha|}$, ce qui permit l'emploi de la méthode de la phase stationaire dans §3 de [4] parce que $5-\gamma < 5/2$. En tenant compte de $n = 2$ et $\sigma = 1$ dans l'hypothèse (H-3) du Théorème 1 de [4], par (4.5) et le Lemme 2.1 de [4] on voit que pour tout k_o il existe deux constantes $c_{k_o} > 0$ et $C_{k_o}'' > 0$ telles que

$$(4.6) \qquad |\widehat{h_{k_o} u_1}(t_o, \zeta_{k_o})| \geq (c_{k_o} - C_{k_o}'' n_{k_o}^{-1/2}) |\zeta_{k_o}|^{-N_k - 14/5},$$

où $\zeta_{k_o} = n_{k_o}^5 \xi^{k_o}$. Ici, on a utilisé (3.9) et le fait que $J'_{\mu_k} = (1,2,\ldots,1)$ au cas où μ_k pair (voir la Proposition 2.1).

Si l'on choisit la suite $\{n_k\}$ telle que

$$(4.7) \qquad c_k - C_k n_k^{-1/2} \geq c_k/2 ,$$

pour la preuve de (3.10) il reste de montrer que pour tout k il existe une constante $c > 0$ indépendante de k telle que

$$(4.8) \qquad c_k = c^{N_k} N_k^{3N_k} .$$

En tenant compte la formule (3.38) de [4] et la Remarque 1 de §3 de [4], il suffit de montrer que pour tout k il existe une $c_o > 0$ indépendante de k telle que

$$(4.9) \qquad |\det(\partial_{t_p} \partial_{t_q} d_{J'_{\mu_k}}(t'_{\mu_k}))| \geq c_o^{\mu_k} \mu_k^{-3\mu_k} ,$$

(voir (3.41) de [4]) ,

où on a employé $y_o = 0$ et $N_k = \mu_k/2$. Cette inégalité sera démontrée dans §5.

5. Démonstration de la Proposition 2.1.

Pour λ_j ($j = 1, 2$) définies par (2.2) posons

$$p_j = \tau + \lambda_j(t,x,\xi) \quad (j = 1,2) .$$

Alors p_j satisfait à la condition

(N.5) $$H_{j_1} H_{j_2} \cdots H_{j_5} p_{j_6} = 0 \quad (j_k = 1, 2),$$

où l'on désigne par H_j le champ Hamiltonian de p_j

$$H_j = \sum_k \left(\frac{\partial p_j}{\partial \xi} \frac{\partial}{\partial x_k} - \frac{\partial p_j}{\partial x_k} \frac{\partial}{\partial \xi_k}\right), \; x_o = t \; , \; \xi_o = \tau \; .$$

Les conditions avec (N.m) avec des entiers $m \geq 1$ ont déjà été définies dans l'Introduction de [3], à la quelle on référe pour le motif de la définition et les résultats sous ces conditions avec $m \leq 3$. De la même façon dans la preuve du Lemme 2.1 de [3] on a

Lemme 5.1. Si la condition (N.5) est vérifiée, alors $H_{j_1} \cdots H_{j_4} p_{j_5}$ *($j_k = 1, 2$) sont constantes le long de toute trajectoire.*

Par ce lemme on voit le long de toute trajectoire partant de ρ_μ

(5.1)
$$H_1^4 p_2 = H_2^4 p_1 = H_1^2 H_2^2 p_1 = H_2^2 H_1^2 p_2 = 24$$
$$H_2 H_1^3 p_2 = H_1 H_2^3 p_1 = H_1 H_2 H_1^2 p_2 = H_2 H_1 H_2^2 p_1 = -24 \; .$$

Soit $\{(t,(x(t),\xi(t)); \; t \in [0, t_o]\} \subset R_t \times T^*(R^4)$ une trajectoire $C(J_\nu, \mathrm{tt}_\nu, \rho_\mu)$. Alors, $H_1^3 p_2(t,x(t),\xi(t))$, $-H_2^3 p_1(t,x(t),\xi(t))$ et $H_1 H_2^2 p_1(t,x(t),\xi(t))$ ($\equiv -H_2 H_1^2 p_2(t,x(t),\xi(t))$, ce qui découle de l'inégalité de Jacobi) sont égales à $f(t)$ une fonction continue et linèaire partiellement telle que $f'(t)$ sur (t_q , t_{q-1}) est 24 ou -24 selon $j_q = 1$ ou 2 et $f(0) = 0$. En effet, au

moyen de (5.1) on peut voir, par exemple, que $d(H_1^3 p_2)/dt$ sur (t_q , t_{q-1}) est égale à $H_1^4 p_2 = 24$ ou $H_2 H_1^3 p_2 = -24$ selon $j_q = 1$ ou 2 . Evidemment $H_1^3 p_2(0,\rho_\mu) = 0$ et il s'ensuit de (N.5) que $d^2(H_1^3 p_2)/dt^2 = 0$.

En outre, il s'ensuit de (N.5) et (5.1) que $H_1^2 p_2(t,x(t),\xi(t))$ et $H_2^2 p_1(t,x(t),\xi(t))$ sont égales à $m(t)$ une fonction continue et quadratique partiellement telle que $m(0) = 0$, $m'' = 24$ et m' sur (t_q , t_{q-1}) soit égale à f ou $-f$ selon $j_q = 1$ ou 2 .

Si l'on pose

$$F(t, J_\nu, \tt_\nu, \rho_\mu) = (\lambda_1 - \lambda_2)(t,x(t),\xi(t)) \tag{5.2}$$

pour la trajectoire $C(J_\nu, \tt_\nu, \rho_\mu)$, alors F est une fonction continuement différentiable d'ordre 1 et du quatrième degré partiellement telle que

$$\begin{cases} F'''' = 24 \text{ ou } -24 , \quad F''' = -f \text{ ou } f , \\ F'' = -m \text{ ou } m \text{ sur } (t_q , t_{q-1}) \quad \text{selon } j_q = 1 \text{ ou } 2 , \end{cases} \tag{5.3}$$

parce que $d^j F/dt^j$ $(j = 1,2,\ldots)$ sont égales à $H_1^j(p_1 - p_2) = -H_1^j p_2$ ou $H_2^j(p_1 - p_2) = H_2^j p_1$ sur (t_q , t_{q-1}) selon $j_q = 1$ ou 2 et que on a l'inégalité $H_1 p_2 = -H_2 p_1$.

En notant que $H_1 p_2(0, \rho_\mu) = 0$ et $(\lambda_1 - \lambda_2)(0,\rho_\mu) = \beta^4$ $(\beta = t_o/2\mu)$ on obtient

$$\begin{aligned} F_-(t) &\equiv F(t; (1),\emptyset,\rho_\mu) = \beta^4 - t^4 , \\ F_+(t) &\equiv F(t; (2),\emptyset,\rho_\mu) = \beta^4 + t^4 \end{aligned} \tag{5.4}$$

(voir Fig. 2),

où l'on désigne par $C((1), \emptyset,\rho_\mu)$ et $C((2), \emptyset,\rho_\mu)$ deux courbes biractéristiques par rapport à λ_1 et λ_2 , respectivement. De plus,

pour $r = 1,\ldots,\mu$ on a

$$F_r(t) \equiv F(t; J'_r, \mathrm{tt}'_r, \rho_\mu) = (-1)^{r-q}(\beta^4 - (t - 2(r-q)\beta)^4)$$

(5.5)

$$\text{sur } t \in [t^r_q, t^r_{q-1}] \quad (q = 1,\ldots,r+1,\ t^r_{r+1} = 0,\ t^r_0 = 0),$$

(voir Fig. 2).

Grace à (5.3)-(5.5) on obtient la première partie de la Proposition 2.1. L'hypothèse (H-1) du Théorème 1 de [4] découle de (5.5). Comme on a obtenu la Proposition 5.2 de [3], en tenant compte de (5.3)-(5.5) on obtient (5.3) de [3] , ce qui montre que l'hypothèse (H-3) du Théorème 1 de [4] est vérifiée avec $\sigma < 1$ parce que $c_0\varepsilon < \varepsilon^\sigma$ pour petit $\varepsilon < 1$.

Avant de constater l'hypothèse (H-2) du Théorème 1 de [4], on va démontrer la première partie du Théorème 2.2. De la même façon à la fin de [3], en tenant compte de $(\lambda_1 - \lambda_2)(0,\rho_0) = 0$ ($\rho_0 = (0,\eta_0)$) on a par (5.3)

$$\overline{\bigcup_k \Gamma^k(t_0 ; \tilde{K})} = \overline{\Gamma^0(t_0 ; \tilde{K})} = \Gamma^0(t_0 ; \tilde{K}) \tag{5.6}$$

pour $\tilde{K}$ défini par (2.4). On obtient

$$\delta_\infty \notin \Gamma^0(t_0 ; \tilde{K}), \tag{5.7}$$

parce que $\lambda_1 - \lambda_2 = 0$ au point (t_0,δ_∞) et elle n'est pas null sur $\Gamma^0(t_0 ; \tilde{K})$.

Maintenant, on va constater l'hypothèse (H-3), c'est à dire, on va montrer

$$|\det(\partial_{t_p}\partial_{t_q} d_{J'_\mu}(\mathrm{tt}'_\mu)| \geq c_0^\mu \mu^{-3\mu} \tag{4.9)'}$$

pour une constante $c_0 > 0$ indépendante de μ . Posons $\delta_\mu = (x_\mu,\xi_\mu)$.

Au moyen de (3.16) et (3.17) de [4] avec $y_o = 0$, $\eta_0 = \eta_\mu$ et $\xi_0 = \xi_\mu$ on obtient

$$\partial_{t_q} F_{J'_\mu} = (\lambda_{j_q} - \lambda_{j_{q+1}})(t_q, x_q, \xi_q) + ({}^t M_{J'_\mu}^{-1} \eta_\mu - \xi_\mu) \cdot \partial_{t_q} d_{J'_\mu} \quad . \tag{5.8}$$

En différentiant par rapport à t_p et en posant $\tilde{t}_\mu = \tilde{t}'_\mu$ on obtient

$$\begin{aligned} a_{p,q} &\equiv \partial_{t_p} \partial_{t_q} F_{J'_\mu}(\tilde{t}') \\ &= (t_o^3/2\mu^3)\delta_{p,q} + (\partial_{t_p} ({}^t M_{J'_\mu}^{-1} \eta_\mu) \cdot \partial_{t_q} d_{J'_\mu})(\tilde{t}'_\mu) \end{aligned} \tag{5.9}$$

$$\text{si} \quad p \geq q \quad ,$$

où on a utilisé $dF(t; J'_\mu , \tilde{t}'_\mu , \rho_\mu)/dt = (-1)^{p-1} 4\beta^3$ pour obtenir le dérivé du premier terme à droite de (5.8). Ici $\delta_{p,q}$ est le δ de Kronecker. Puisque le calcul direct montre que le deuxième terme à droite de (5.9) est égale à $(-1)^{q-1}$ multiplié par un vecteur indépendant de p , on obtient par (5.9)

$$a_{p,q} + a_{p,q-1} = (t_o/2\mu^3) \ \delta_{p,q} \qquad \text{si} \quad p \geq q \ . \tag{5.10}$$

De plus, le calcul direct donne

$$a_{1,1} = (2\mu-1)/\mu^3 \quad , \quad a_{p-1,p} + a_{p-1,p-1} = 5t_o^3/2\mu^3 \quad , \ 2 \leq p \leq \mu \ . \tag{5.11}$$

Si l'on désigne par T_μ une $\mu \times \mu$ matrice

$$\begin{bmatrix} 1 & 0 & & & \\ 1 & 1 & 0 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & 0 \\ 0 & & & 1 & 1 \end{bmatrix}$$

alors il s'ensuit de (5.10), (5.11) et $a_{p,q} = a_{q,p}$ que

$$T_\mu \ (a_{p,q}) \ {}^t T_\mu = (t_o/2\mu^3)^\mu \ S_\mu .$$

Ici on a désigné par S_q pour $1 \leq q \leq \mu$, $q \times q$ matrice de Jacobi

$$\begin{bmatrix} b_1 & d_1 & & & 0 \\ d_1 & b_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & d_{q-1} \\ 0 & & & d_{q-1} & b_q \end{bmatrix} \qquad \begin{aligned} b_1 &= 4\mu - 2 \ , \quad d_1 = 5 \\ b_q &= 6 \ , \ d_q = 1 \ , \quad q \geq 2. \end{aligned}$$

Si l'on pose $\square_q = \det S_q$ alors on a

$$\square_q = b_q \square_{q-1} - d_{q-1}^2 \square_{q-2} \tag{5.12}$$

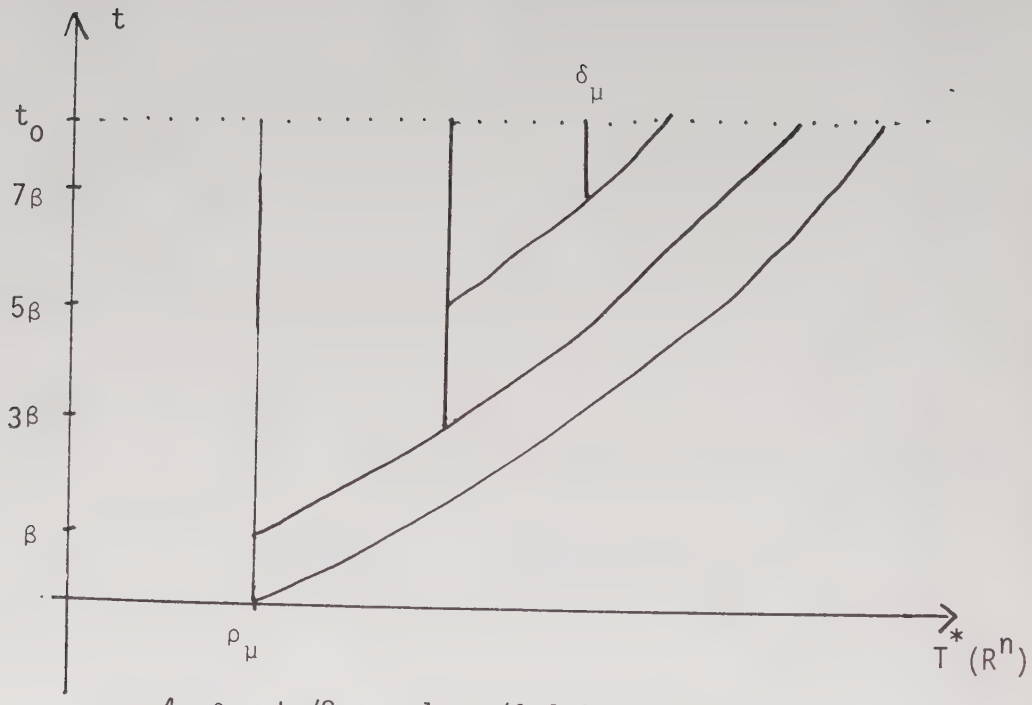
$$= 6\square_{q-1} - \square_{q-2} \qquad \text{si} \quad q \geq 3 \ ,$$

ce qui montre

$$\det(a_{p,q}) = (t_o/2\mu^3)^{\mu}((\alpha_+^{\mu} - \alpha_-^{\mu})(24\mu - 37) \tag{5.13}$$

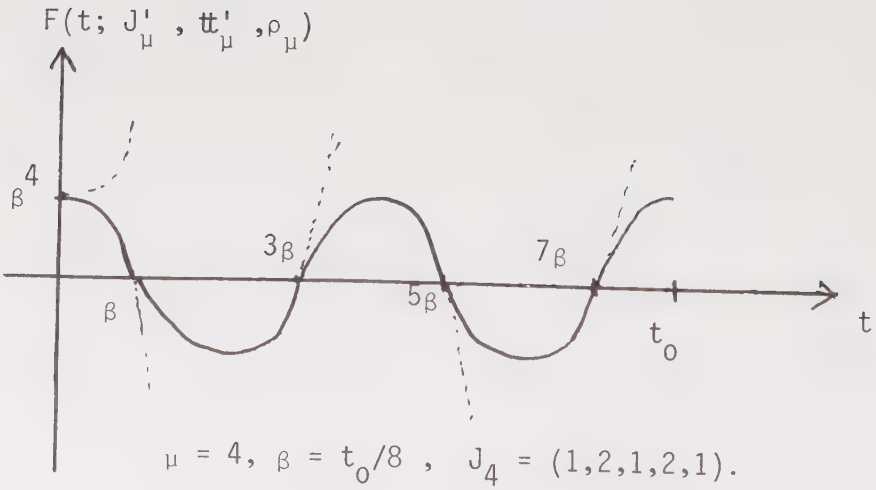
$$+ (\alpha_+^{\mu-2} - \alpha_-^{\mu-2})(2 - 4\mu)) \ ,$$

où $\alpha_{\pm} = 3 \pm 2\sqrt{2}$. Donc, on obtient (4.9)' pour une $c_o > 0$ indépendante de μ.



$\mu = 4$, $\beta = t_o/8$, $J_4 = (1,2,1,2,1)$.

Fig. 1



$\mu = 4$, $\beta = t_o/8$, $J_4 = (1,2,1,2,1)$.

Fig. 2

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On the hypoellipticity for infinitely degenerate semi-elliptic operators

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§ 0. Introduction.

In this paper we shall study hypoellipticity for a partial differential operator of the form

$$P=a(x, y, D_x)+g(x)b(x, y, D_y) \quad \text{in} \quad R^n=R_x^{n_1}\times R_y^{n_2}, \tag{0.1}$$

where $a(x, y, D_x)$ and $b(x, y, D_y)$ are strongly elliptic operators of order $2l$ and $2m$ with respect to x and y, respectively, and $g(x)$ is a smooth non-negative function with a zero point of infinite order at $x=0$ in $R_x^{n_1}$. The operator of the form $(-\Delta_x)^l+g(x)(-\Delta_y)^m$ is a typical example.

Our main theorem is roughly stated as follows: Assume that $b(x, y, D_y)$ is of second order (, but $a(x, y, D_x)$ is not necessarily of second order). Then we have the statement:

$$(*) \qquad \text{“}u\in\mathcal{D}'(\Omega),\quad Pu\in H_s^{loc}(\Omega)\Rightarrow u\in H_s^{loc}(\Omega)\text{”} \quad \text{for any } \Omega\subset R^n,$$

and therefore P is hypoelliptic in R^n. When $b(x, y, D_y)$ is of higher order $\geqq 4$, we set the following condition on $g(x)$.

Condition (G).

$$|\partial_x^\beta g(x)|\leqq C_\beta g(x)^{1-\sigma|\beta|} \quad \text{in a neighborhood of} \quad x=0$$

for a fixed σ $(0<\sigma<\{2(m+l(m-1))\}^{-1})$. Then, we have the statement (*) in this case, too. (Such a σ is determined from Propositions 5.1 and 5.2. See Remark of Proposition 5.2.)

When $g(x)$ has a zero point of finite order, fairly complete results have been obtained by Hörmander [8], [9], Grushin [6], [7], Beals [1], Y. Kato [11], Kumano-go-Taniguchi [15], Taniguchi [17], Tsutsumi [19], etc. In such case except [17] we have the stronger result than (*), that is, the statement

$$(**) \qquad \text{“}u\in\mathcal{D}'(\Omega),\quad Pu\in H_s^{loc}(\Omega)\Rightarrow u\in H_{s+\sigma_0}^{loc}(\Omega)\text{”} \quad \text{for any} \quad \Omega\subset R^n$$

holds for some positive number σ_0. It should be noted that we can no longer expect the statement (**) for the operator of the form (0.1) when $g(x)$ has a zero point of infinite order (see Theorem 1.2).

As an example which does not satisfy the sufficient condition for hypoellipticity given by Hörmander [9], V. S. Fedii [5] considered an operator $P=\partial_x^2+\phi(x)^2\partial_y^2$ where $\phi(x)$ satisfies $\phi(x)>0$ for $x\neq 0$ and $\phi^{(j)}(0)=0$ for any j, and he proved the statement (*) for that operator, improving the criterion given by Treves [18]. Remark that his result can be obtained by setting $g(x)=\phi(x)^2$.

The plan of this paper is as follows. In Section 1 we state our main result (Theorem 1.1). In Section 2 we introduce a notion "weakly-elliptic" for an operator P concerning sufficient conditions for hypoellipticity given in [5], [16], [18], and we prove that the statement (*) holds for the weakly-elliptic operators (, so P is necessarily hypoelliptic). In Sections 3 and 4 we prove that P of the form (0.1) is weakly-elliptic. Section 5 is devoted to the proof of Lemma 4.1, which plays an important role in Section 4.

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§ 1. Notations and main result.

For a multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ of non-negative integers α_j $(j=1, \cdots, n)$ and a point $x=(x_1, \cdots, x_n)\in R^n$ we use the notations:

$$|\alpha|=\alpha_1+\cdots+\alpha_n, \quad \alpha!=\alpha_1!\cdots\alpha_n!, \quad x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}, \quad \partial_{x_j}=\frac{\partial}{\partial x_j}, \quad D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n},$$

$$D_{x_j}=-i\frac{\partial}{\partial x_j}, \quad \langle x\rangle=\{1+|x|^2\}^{1/2}.$$

Let $C^\infty(\Omega)$ denote the set of infinitely differentiable functions in Ω and let $C_0^\infty(\Omega)$ denote the set of $C^\infty(\Omega)$-functions with compact support, where Ω is an open set in R^n. For a compact set K of Ω, $C_0^\infty(K)$ denotes the set of $C^\infty(\Omega)$-functions whose supports are contained in K. Let $\mathcal{B}(\Omega)$ denote the set of $C^\infty(\Omega)$-functions whose derivatives of any order are all bounded in Ω, and let $\mathcal{D}'(\Omega)$ denote the set of distributions in Ω. $\mathcal{E}'(K)$ is the set of distributions with support in K. $C^\infty(R^n)$, $C_0^\infty(R^n)$, $\mathcal{B}(R^n)$, $\mathcal{D}'(R^n)$ are often denoted simply by C^∞, C_0^∞, $\mathcal{B}$, $\mathcal{D}'$, respectively. $\mathcal{S}=\mathcal{S}(R^n)$ denotes the Schwartz space of rapidly decreasing functions and $\mathcal{S}'$ denotes its dual space. For $u\in\mathcal{S}_x$ the Fourier transform $\hat{u}(\xi)=\mathcal{F}[u](\xi)$ is defined by

$$\mathcal{F}[u](\xi)=\int e^{-ix\cdot\xi}u(x)dx, \quad x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n.$$

and then, for $\hat{u}(\xi)\in\mathcal{S}_\xi$, the inverse Fourier transform $\overline{\mathcal{F}}[\hat{u}]$ is defined by

$$\overline{\mathcal{F}}[\hat{u}](x)=\int e^{ix\cdot\xi}\hat{u}(\xi)\,d\!\!\!^{-}\xi, \qquad d\!\!\!^{-}\xi=(2\pi)^{-n}d\xi.$$

For real s we define the Sobolev space H_s as the completion of $\mathcal{S}$ in the norm

$$\|u\|_s=\left\{\int\langle\xi\rangle^{2s}|\hat{u}(\xi)|^2\,d\!\!\!^{-}\xi\right\}^{1/2}.$$

$H_s^{loc}(\Omega)$ denotes the space $\{u\in\mathcal{D}'(\Omega)\,;\ \phi u\in H_s \text{ for } \phi\in C_0^\infty(\Omega)\}$. For ϕ, $\psi\in C_0^\infty(\Omega)$ we write $\phi\subset\subset\psi$ when $\psi(x)\equiv1$ in a neighborhood of $\operatorname{supp}\phi$.

Now we shall study the hypoellipticity of a differential operator P of the form

$$P=a(x,y,D_x)+g(x)b(x,y,D_y) \qquad \text{in } R^n=R_x^{n_1}\times R_y^{n_2} \tag{0.1}$$

under the following conditions:

1°) $a(x,y,D_x)$ is a differential operator of order $2l$ with C^∞ coefficients and satisfies for large $|\xi|$

$$\operatorname{Re}\,a(x,y,\xi)\geqq C_1|\xi|^{2l} \qquad (C_1>0). \tag{1.1}$$

2°) $b(x,y,D_y)$ is a differential operator of order $2m$ with C^∞ coefficients and satisfies for large $|\eta|$

$$\operatorname{Re}\,b(x,y,\eta)\geqq C_2|\eta|^{2m} \qquad (C_2>0). \tag{1.2}$$

3°) $g(x)$ belongs to $C^\infty(R_x^{n_1})$, $g(x)>0$ for $x\neq0$, and $\partial_x^\beta g(0)=0$ for any β.

THEOREM 1.1. *Let P satisfy Conditions* 1°), 2°) *and* 3°). *Then, we obtain the following two assertions:*

i) *If $m=1$ (, i.e., $b(x,y,D_y)$ is of second order), then the statement*

(*) $\quad$ *"$u\in\mathcal{D}'(\Omega)$, $Pu\in H_s^{loc}(\Omega)\Rightarrow u\in H_s^{loc}(\Omega)$" for ony $\Omega\subset R^n$*

holds. Therefore, P is hypoelliptic.

ii) *In case $m\geqq2$, we add the following condition* (G) *for $g(x)$. Then we have the statement* (*).

Condition (G). *For any β there exists a constant C_β such that*

$$|\partial_x^\beta g(x)|\leqq C_\beta g(x)^{1-\sigma|\beta|} \qquad \text{in a neighborhood of } x=0. \tag{1.3}$$

Here σ is a fixed positive number such that

$$0<\sigma<\{2(m+l(m-1))\}^{-1}. \tag{1.4}$$

REMARK 1. It is easy to see that in general any non-negative function $g(x)$ always satisfies the inequality (1.3) with $\sigma=1/2$. In fact, for $|x|\leqq 1$, $-1\leqq t\leqq 1$ and $M=\max_{j,\,|x|\leqq 2}|\partial_{x_j}^2 g(x)|$ the inequality

$$0\leqq g(x+te_j)\leqq g(x)+t\partial_{x_j}g(x)+\frac{t^2}{2}M$$

holds, and so, setting

$$t=-\operatorname{sign}(\partial_{x_j}g(x))\sqrt{g(x)}\,/\sqrt{g(x)^2+(M/2)^2}$$

we have

$$|\partial_{x_j}g(x)|\leqq C\sqrt{g(x)}\quad\text{on}\quad\{|x|\leqq 1\}.$$

REMARK 2. We see that the function $e^{-(1/|x|)}$ satisfies Condition 3°) and the inequality (1.3) with arbitrary positive σ. The function

$$f(x)=e^{-(1/|x|)}\sin^2\frac{1}{|x|}+e^{-(1/|x|)^2}$$

does not satisfy the inequality (1.3) for any σ $(<1/2)$, though it satisfies Condition 3°). Because we have for integer k

$$\partial_{x_j}^2 f(x)=0(e^{-\pi k}),\quad f(x)=0(e^{-\pi^2 k^2}),\quad |x|=(\pi k)^{-1}\qquad\text{as}\quad k\to\infty.$$

We note that f is not expressed in the form $f(x)=\phi(x)^2$ for any non-negative C^∞-function ϕ.

As stated in Introduction, if we replace the condition "$\partial_x^\beta g(0)=0$ for any β" in Condition 3°) by "$\partial_x^{\beta_0}g(0)\neq 0$ for some β_0", we can expect a stronger statement than (*):

(**) "$u\in\mathcal{D}'(\Omega)$, $Pu\in H_s^{loc}(\Omega)\Rightarrow u\in H_{s-\sigma_0}^{loc}(\Omega)$" for any $\Omega\subset R^n$,

for some $\sigma_0>0$. It is impossible to replace (*) by (**) in Theorem 1.1, in general. To explain this fact we prove the following

THEOREM 1.2. *Let $P=p(x, y, D_x, D_y)$ be a differential operator of the form (0.1). We assume that there exists some $\sigma_0>0$ such that*

$$\|u\|_{\sigma_0}^2\leqq C(\|Pu\|_0^2+\|u\|_0^2),\qquad u\in C_0^\infty(\Omega) \tag{1.5}$$

for some constant C when Ω is an open set (containing $\{0\}$) of $R_{x,y}^n$. Then, there exists some β_0 such that $\partial_x^{\beta_0}g(0)\neq 0$.

To prove this theorem we use the following theorem (see Hörmander [8]).

THEOREM 1.3. *Let $P=p(x, D_x)$ be a differential operator of order m with coefficients in $C^\infty(R_x^n)$. Suppose that, for a fixed $\sigma_0>0$ and an open set Ω of R_x^n, we have the estimate*

$$\|u\|_{\sigma_0}^2 \leqq C_1(\|Pu\|_0^2+\|u\|_0^2), \qquad u\in C_0^\infty(\Omega)$$

for some constant C_1. Then, for any integer $N>0$, any θ $(0<\theta\leqq 1/2)$ and any compact set K of Ω we have the estimate

$$\begin{aligned}(1.6)\qquad &\int\langle\xi\rangle^{2\sigma_0}|\psi(y)|^2dy\\ &\leqq C_2\Big\{\int\Big|\sum_{|\alpha+\beta|<N} p_{(\beta)}^{(\alpha)}(x,\xi)\langle\xi\rangle^{\theta(|\alpha|-|\beta|)}(iy)^\beta D_y^\alpha\psi(y)/(\alpha!\,\beta!)\Big|^2dy\\ &\quad+\langle\xi\rangle^{2m-2\theta N}\sum_{|\alpha+\beta|\leqq N+m}\int|y^\beta D_y^\alpha\psi(y)|^2dy\Big\},\\ &\qquad (x,\xi)\in K\times R_\xi^n,\quad \psi(y)\in C_0^\infty(R_y^n),\end{aligned}$$

for a constant $C_2=C_2(K)$. Here $p_{(\beta)}^{(\alpha)}(x,\xi)=\partial_\xi^\alpha D_x^\beta p(x,\xi)$.

PROOF OF THEOREM 1.2. Assume that $\partial_x^\beta g(0)=0$ for any β. Then, applying Theorem 1.3 to (1.5), that is, setting $(x, y)=(0, 0)$ and $(\xi, \eta)=(0, \eta)$ for $p(x, y, D_x, D_y)$, we have from (1.6)

$$\begin{aligned}&\langle\eta\rangle^{2\sigma_0}\|\psi(z)\|^2\\ &\quad\leqq C'\{\|\sum_{|\alpha+\beta|<N} p_{(\beta)}^{(\alpha)}(0,0,0,\eta)\langle\eta\rangle^{\theta(|\alpha|-|\beta|)}(iz)^\beta\\ &\qquad\times D_z^\alpha\psi(z)/(\alpha!\,\beta!)\|^2+\langle\eta\rangle^{2M-2\theta N}\sum_{|\alpha+\beta|\leqq N+M}\|z^\beta D_z^\alpha\psi(z)\|^2\},\\ &\qquad\psi(z)\in C_0^\infty(R_z^n),\quad R_z^n=R_x^{n_1}\times R_y^{n_2},\end{aligned}$$

where $M=\max(2l, 2m)$. Fix $\psi(z)\not\equiv 0$ and θ so that $2\theta l<\sigma_0$. Then, taking N so large that $M-\theta N<0$ and noting that when $|\alpha|>2l$ (=the order of $a(x, y, \xi)$ in ξ) $p_{(\beta)}^{(\alpha)}(0, 0, 0, \eta)=0$ for any β, we have

$$\langle\eta\rangle^{2\sigma_0}\leqq C''\langle\eta\rangle^{4\theta l} \qquad \text{for any} \quad \eta,$$

which is a contradiction. Q. E. D.

§ 2. Weakly-elliptic operator and hypoellipticity.

DEFINITION 2.1. A differential operator $P=p(x, D_x)$ of order m with coefficients in $C^\infty(\Omega)$ is called 'weakly-elliptic' (or simply 'w-elliptic') in Ω if P satisfies the following conditions:

I) For any compact set K of Ω and any $N>0$ there exists a constant $C_1=C_1(K, N)$ such that

$$\|u\|_0 \leqq C_1(\|Pu\|_0+\|u\|_{-N}), \quad u\in C_0^\infty(K). \tag{2.1}$$

II) For any compact set K of Ω, and β ($|\beta|\neq 0$), any $\mu>0$ and any $N>0$ there exists a constant $C_2=C_2(K, \beta, \mu, N)$ such that

$$\|P_{(\beta)}u\|_{-|\beta|} \leqq \mu\|Pu\|_0+C_2\|u\|_{-N}, \quad u\in C_0^\infty(K), \quad (|\beta|\neq 0) \tag{2.2}$$

where $p_{(\beta)}(x, \xi)=D_x^\beta p(x, \xi)$.

III) For any $x_0\in\Omega$ and any neighborhood U of x_0 there exists $\phi(x)\in C_0^\infty(U)$ such that

$$\phi(x)\equiv 1 \quad \text{in some neighborhood of } x_0$$

and the estimate

$$\|[P, \phi]u\|_s \leqq C_3(K, s, N, \phi)(\|Pu\|_{s-\kappa}+\|u\|_{-N}), \quad u\in C_0^\infty(K) \tag{2.3}$$

holds for any compact set K, any real s and any $N>0$, where $\kappa>0$ may depend on x_0, and $[P, \phi]=P\phi-\phi P$.

REMARK. We see that many hypoelliptic operators are w-elliptic. For instance semi-elliptic operators, subelliptic operators in Egorov [3], [4] and hypoelliptic operators treated in Hörmander [10] are w-elliptic. But, in general, hyperbolic operators are not w-elliptic, since the estimates II) and III) do not hold.

THEOREM 2.2. *Let a differential operator P be w-elliptic in Ω. Then, for any open set $\Omega'\subset\Omega$ we have*

$$v\in\mathcal{D}'(\Omega'), \quad Pv\in H_s^{\mathrm{loc}}(\Omega') \Rightarrow v\in H_s^{\mathrm{loc}}(\Omega').$$

REMARK 1. This theorem is a re-formation of theorems given by Fedii [5], Oleinik-Radkevich [16] and Treves [18]. It seems that the assumptions of this theorem does not include the following estimate (, which is derived from the hypotheses of their theorems): "For any real s, any $\mu>0$ and any $N>0$ there exists a constant C such that

$$\|P^{(j)}u\|_s \leqq \mu\|Pu\|_s + C_\mu\|u\|_{-N}, \quad u\in C_0^\infty(K),$$

where $p^{(j)}(x, \xi)=\partial_{\xi_j}p(x, \xi)$." We do not know whether the above estimate holds for the operators P in Theorem 1.1, when the order of $a(x, y, D_x)$ is not less than 4.

REMARK 2. Theorem 2.2 will be proved by showing that, for any fixed point $x_0 \in \Omega'$, there exist ϕ, $\psi \in C_0^\infty(\Omega')$ such that $\phi(x) \equiv 1$ in a neighborhood of x_0 and $\phi \subset\subset \psi$, and for some large $N>0$ and a constant C the estimate

$$\|\phi v\|_s \leqq C(\|\psi P v\|_s + \|\psi v\|_{-N}) \tag{2.4}$$

holds. Therefore, by modifying the coefficients of P outside of $\operatorname{supp}\psi$ we may assume without loss of generality that all the coefficients of P belong to $\mathcal{B}(R^n)$.

Before the proof of Theorem 2.2 we define pseudodifferential operators and state several propositions without proofs (see [2], [8], [12], [13] and [14]). Let $\lambda(\xi)$ be a basic weight function in R^n with the following conditions:

$$A_0^{-1}\langle\xi\rangle^{\sigma'} \leqq \lambda(\xi) \leqq A_0\langle\xi\rangle \qquad (A_0>0,\ \sigma'>0), \tag{2.5}$$

$$|\lambda^{(\alpha)}(\xi)| \leqq A_\alpha \lambda(\xi)^{1-|\alpha|}. \tag{2.6}$$

By using this fixed function $\lambda(\xi)$ we define a class of pseudodifferential operators.

DEFINITION 2.3. We say that the C^∞-function $p(x, \xi)$ in $R^{2n}_{x,\xi}$ belongs to $S^m_{\lambda,\rho,\delta}$ $(-\infty<m<\infty,\ \delta=(\delta_1, \cdots, \delta_n),\ \rho=(\rho_1, \cdots, \rho_n),\ 0 \leqq \delta_j < \rho_j \leqq 1,\ j=1, \cdots, n)$, when for any multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_n)$ we have

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_{\alpha,\beta}\lambda(\xi)^{m-\rho\cdot\alpha+\delta\cdot\beta} \qquad \text{on} \quad R^{2n}, \tag{2.7}$$

where

$$p^{(\alpha)}_{(\beta)}(x, \xi) = \partial_\xi^\alpha D_x^\beta p(x, \xi), \quad \rho\cdot\alpha = \sum_{j=1}^n \rho_j\alpha_j, \quad \delta\cdot\beta = \sum_{j=1}^n \delta_j\beta_j.$$

Then, the pseudodifferential operator $P=p(x, D_x)$ (denoted by $P \in S^m_{\lambda,\rho,\delta}$) with the symbol $p(x, \xi)$ is defined by

$$Pu = \int e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi)d\xi, \quad u \in \mathcal{S}. \tag{2.8}$$

We often denote the symbol of $P=p(x, D_x)$ by $\sigma(P)$.

REMARK 1. When $\rho=(\rho_0, \cdots, \rho_0)$, $\delta=(\delta_0, \cdots, \delta_0)$, we denote $S^m_{\lambda,\rho,\delta}$, $S^m_{\lambda,\rho,\delta}$ by $S^m_{\lambda,\rho_0,\delta_0}$, $S^m_{\lambda,\rho_0,\delta_0}$, respectively. Furthermore when $\lambda=\langle\xi\rangle$, $\rho_0=1$, $\delta_0=0$, we often write $S^m_{\lambda,\rho_0,\delta_0}=S^m$, $S^m_{\lambda,\rho_0,\delta_0}=S^m$. A differential operator of order m with coefficients in $\mathcal{B}(R^n)$ is clearly the element of S^m.

2. $S^m_{\lambda,\rho,\delta}$ is Fréchet space provided with semi-norms

$$|p|^{(m)}_l = \max_{|\alpha+\beta| \leq l} \sup_{R^{2n}} \{|p^{(\alpha)}_{(\beta)}(x, \xi)|\lambda(\xi)^{-m+\rho\cdot\alpha-\delta\cdot\beta}\}.$$

We say that the subset B of $S^m_{\lambda,\rho,\delta}$ is bounded when $\sup_{p\in B}\{|p|^{(m)}_l\} < \infty$ for any l.

3. We set $S^{-\infty}_{\lambda,\rho,\delta}=\bigcap_{-\infty<m<\infty} S^m_{\lambda,\rho,\delta}$. It is easy to see that $S^{-\infty}_{\lambda,\rho,\delta}=S^{-\infty}$.

PROPOSITION 2.4. *Let* $P_j=p_j(x, D_x)\in S^{m_j}_{\lambda,\rho,\delta}$. *Then, we have* $P_1P_2\in S^{m_1+m_2}_{\lambda,\rho,\delta}$ *and*

$$\sigma(P_1P_2)-\sum_{|\alpha|<N}\frac{1}{\alpha!}p_1^{(\alpha)}(x,\xi)p_{2(\alpha)}(x,\xi)\in S^{m_1+m_2-\tau_0 N}_{\lambda,\rho,\delta} \tag{2.9}$$

where $\tau_0=\min_j(\rho_j-\delta_j)$. *Furthermore, we have*

$$[P_1, P_2]-\sum_{0<|\alpha+\beta|<N}\frac{(-1)^{|\beta|}}{\alpha!\beta!}P_{2(\beta)}^{(\alpha)}P_{1(\alpha)}^{(\beta)}\in S^{m_1+m_2-\tau_0 N}_{\lambda,\rho,\delta}. \tag{2.10}$$

We define $\|u\|_{s,\lambda}$ by

$$\|u\|_{s,\lambda}=\left(\int\lambda(\xi)^{2s}|\hat{u}(\xi)|^2d\xi\right)^{1/2}\qquad\text{for}\quad u\in\mathcal{S}.$$

When $\lambda=\langle\xi\rangle$, we denote $\|u\|_{s,\lambda}=\|u\|_s$.

PROPOSITION 2.5. *Let* $P=p(x, D_x)\in S^m_{\lambda,\rho,\delta}$. *Then for any real* s *there exist* C *and* l *depending on* m *and* s *such that*

$$\|Pu\|_{s,\lambda}\leqq C|p|_l^{(m)}\|u\|_{s+m,\lambda}\qquad \textit{for}\quad u\in\mathcal{S}. \tag{2.11}$$

PROPOSITION 2.6. *Let* $P=p(x, D_x)\in S^m_{\lambda,\rho,\delta}$ *and let* ϕ *and* $\psi\in\mathcal{B}$ *satisfy* $\operatorname{dis}(\operatorname{supp}\phi, \operatorname{supp}\psi)\geqq C_0>0$. *Then,* $\phi P\psi\in S^{-\infty}$.

PROPOSITION 2.7. *Let* $p(x, \xi)\in S^m_{\lambda,\rho,\delta}$ *satisfy the following conditions (which is called (H)-condition):*

i) *There exist* $C_0>0$ *and some real* m' *such that*

$$|p(x,\xi)|\geqq C_0\lambda(\xi)^{m'}\qquad \textit{for large}\quad |\xi|. \tag{2.12}$$

ii) *For any* α, β *there exists a constant* $C_{\alpha,\beta}$ *such that*

$$|p_{(\beta)}^{(\alpha)}(x,\xi)/p(x,\xi)|\leqq C_{\alpha,\beta}\lambda(\xi)^{\delta\cdot\beta-\rho\cdot\alpha}\qquad \textit{for large}\quad |\xi|. \tag{2.13}$$

Then, there exists a parametrix $Q=q(x, D_x)\in S^{-m'}_{\lambda,\rho,\delta}$ *such that* $I\equiv QP\equiv PQ$ $(\operatorname{mod} S^{-\infty})$, *that is,*

$$I=QP+K=PQ+K'\qquad \textit{for some } K \textit{ and } K'\in S^{-\infty}. \tag{2.14}$$

Furthermore, we can write

$$\begin{cases} Q=Q_0Q_1 \\ \textit{for}\quad Q_0\in S^{-m'}_{\lambda,\rho,\delta}\quad \textit{defined by}\quad \sigma(Q_0)=q_0(x,\xi)=p(x,\xi)^{-1}\quad \textit{for large}\quad |\xi|, \\ \textit{and some}\quad Q_1\in S^0_{\lambda,\rho,\delta}. \end{cases} \tag{2.15}$$

PROOF. Note that $q_0^{(\alpha)}{}_{\beta}(x, \xi)$ consists of the linear combination of symbols of the form

$$(p^{-1}p^{(\alpha^1)}_{(\beta^1)})\cdots(p^{-1}p^{(\alpha^k)}_{(\beta^k)})p^{-1} \qquad \text{for large } |\xi|$$

$$(\alpha^1+\cdots+\alpha^k=\alpha,\ \beta^1+\cdots+\beta^k=\beta,\ k\leqq|\alpha|+|\beta|).$$

Then, from (2.12) and (2.13) we have $q_0(x, \xi)\in S^{-m'}_{\lambda,\rho,\delta}$, so $Q_0\in S^{-m'}_{\lambda,\rho,\delta}$. Similarly, using (2.9) in Proposition 2.4, from (2.12) and (2.13) it is easy to check $\sigma(I-PQ)\equiv S^{-\tau_0}_{\lambda,\rho,\delta}$, where $\tau_0=\min_j(\rho_j-\delta_j)$. Set $R=I-PQ_0$. Then, since $R^j\in S^{-\tau_0 j}_{\lambda,\rho,\delta}$, there exists $Q_1\equiv S^0_{\lambda,\rho,\delta}$ such that $Q_1\sim\sum_{j=0}^{\infty}R^j$ in the sense

$$Q_1-\sum_{j=0}^{N-1}R^j\in S^{-\tau_0 N}_{\lambda,\rho,\delta}$$

holds for any N (see [8]). Asymptotically, we have

$$(I-R)Q_1\sim(I-R)\sum_{j=1}^{\infty}R^j\sim I.$$

Consequently we obtain

$$PQ_0Q_1=(I-R)Q_1=I+K' \qquad \text{for some } K'\in S^{-\infty}.$$

Hence, we see that $Q=Q_0Q_1\in S^{-m'}_{\lambda,\rho,\delta}$ is a right parametrix. Similarly, checking $\sigma(I-Q_0P)$, we have a left parametrix Q' such that $Q'P\equiv I \pmod{S^{-\infty}}$. By means of the equality

$$Q'\equiv Q'(PQ)\equiv(Q'P)Q\equiv Q \qquad (\mathrm{mod}\, S^{-\infty}),$$

we have $QP\equiv I \pmod{S^{-\infty}}$, that is, $QP=I+K$, $K\in S^{-\infty}$. Q.E.D.

REMARK. Let $p(x, \xi)\in S^m_{\lambda,\rho,\delta}$ satisfy

$$|p(x, \xi)|\geqq C_0\lambda(\xi)^m \quad (C_0>0) \qquad \text{for large } |\xi|.$$

Then, using $|p^{(\alpha)}_{\beta}(x, \xi)|\leqq C_{\alpha,\beta}\lambda(\xi)^{m+\delta\cdot\beta-\rho\cdot\alpha}$, we get (2.13). Therefore, $p(x, D_x)\in S^m_{\lambda,\rho,\delta}$ has a parametrix $Q\in S^{-m}_{\lambda,\rho,\delta}$.

LEMMA 2.8. *Let $\Lambda_{s,k,\varepsilon}$ denote a pseudodifferential operator with the symbol $\langle\xi\rangle^s(1+\varepsilon\langle\xi\rangle)^{-k}$ for real s, $\varepsilon>0$ and $k\geqq0$. Then, $\Lambda_{s,k,\varepsilon}\in S^{s-k}$, and for any $\varepsilon>0$ there exists a constant C_ε such that*

$$\|\Lambda_{s,k,\varepsilon}u\|_{s'}\leqq C_\varepsilon\|u\|_{s+s'-k}, \quad u\in\mathcal{S}. \tag{2.16}$$

Furthermore, $\{\Lambda_{s,k,\varepsilon}\}$ is a bounded set in S^s, so the estimate

$$\|\Lambda_{s,k,\varepsilon}u\|_{s'}\leqq C_0\|u\|_{s'+s}, \quad u\in\mathcal{S} \tag{2.17}$$

holds for a constant C_0 independent of ε. Moreover, for any α there exists a constant C_α such that

$$\|\Lambda^{(\alpha)}_{s,k,\varepsilon}u\|_0 \leqq C_\alpha \|\Lambda_{s-|\alpha|,k,\varepsilon}u\|_0, \quad u\in\mathcal{S}. \tag{2.18}$$

PROOF. (2.16) and (2.17) are clear. Using the inequality

$$|\partial_\xi^\alpha(\langle\xi\rangle^s(1+\varepsilon\langle\xi\rangle)^{-k})| \leqq C'_\alpha\langle\xi\rangle^{s-|\alpha|}(1+\varepsilon\langle\xi\rangle)^{-k}$$

we obtain (2.18). Q. E. D.

LEMMA 2.9. *Let $\Xi=\{\boldsymbol{a}=(a_0, a_1, \cdots, a_l)\}$ be a set of real vectors with $a_j>0$ $(1\leqq j\leqq l)$ satisfying the following: For any $\mu>0$ there exists some constant C_μ such that*

$$a_j \leqq \mu\sum_{k=0}^{l-1} a_k + C_\mu \sum_{k=j+1}^{l} a_k \quad \text{for any } \boldsymbol{a}\in\Xi$$

$$(j=1, \cdots, l-1).$$

Then, for any $\mu'>0$, we can find a constant $C'_{\mu'}$ such that

$$a_j \leqq \mu' a_0 + C'_{\mu'} a_l \quad \text{for any } \boldsymbol{a}\in\Xi \quad (j=1, \cdots, l-1).$$

PROOF. For any $\mu_j>0$ $(1\leqq j\leqq l-1)$ we have

$$a_j \leqq \mu_j \sum_{k=0}^{l-1} a_k + C_{\mu_j} \sum_{k=j+1}^{l} a_k \quad \text{for any } \boldsymbol{a}\in\Xi$$

for some C_{μ_j}. Multiplying both sides by some $d_j>0$ and taking the sum over $1\leqq j\leqq l-1$, we obtain

$$\sum_{j=1}^{l-1} d_j a_j \leqq \Big(\sum_{j=1}^{l-1}\mu_j d_j\Big)\sum_{k=0}^{l-1} a_k + \sum_{j=0}^{l-1} d_j C_{\mu_j} \sum_{k=j+1}^{l} a_k.$$

Then we have

$$\sum_{j=1}^{l-1} h_j a_j \leqq \Big(\sum_{j=1}^{l-1}\mu_j d_j\Big)a_0 + \Big(\sum_{j=1}^{l-1} d_j C_{\mu_j}\Big)a_l,$$

where

$$h_1 = d_1 - \sum_{k=1}^{l-1}\mu_k d_k, \quad h_j = d_j - \sum_{k=1}^{l-1}\mu_k d_k - \sum_{k=1}^{j-1} d_k C_{\mu_k}$$

$$(j=2, \cdots, l-1).$$

For any $\mu'>0$ we can find $d_j>0$, $\mu_j>0$ such that

$$h_j>1-\mu', \quad \sum_{j-1}^{l-1}\mu_j d_j < \mu' \quad (j=1, \cdots, l-1).$$

(For instance, we can determine $\{d_j, \mu_j\}$ by

$$d_1=1,\ \mu_1=\mu'/2,\ d_2=1+C_{\mu_1}d_1,\ \mu_2 d_2=\mu'/4,\ \cdots,$$

$$d_j=1+\sum_{k=1}^{j-1} C_{\mu_k}\mu_k,\ \mu_j d_j=\mu'/2^j,\ \cdots,\ (j=1,\ \cdots,\ l-1).)$$

Therefore, we have for any $0<\mu'<1$

$$(1-\mu')\sum_{j=1}^{l-1} a_j \leqq \mu' a_0+\Big(\sum_{j=1}^{l-1} d_j C_{\mu_j}\Big)a_l, \qquad \boldsymbol{a}\in\Xi. \qquad \text{Q. E. D.}$$

From now on let P denote a differential operator of order m with coefficients $\in\mathcal{B}(R^n)$.

LEMMA 2.10. *Let P satisfy Condition* II) *of Definition* 2.1. *Then, for any compact set $K\subset\Omega$, any β $(|\beta|\neq 0)$, any real s, any $\mu>0$, $N>0$, $\varepsilon>0$ and $k\geqq 0$ there exists a constant $C=C(K, \beta, s, \mu, N, k)$ independent of ε such that*

$$\|\Lambda_{s-1,\beta|,k,\varepsilon}P_{(\beta)}u\|_0\leqq\mu\|\Lambda_{s,k,\varepsilon}Pu\|_0+C\|u\|_{-N},\quad u\in C_0^\infty(K). \tag{2.19}$$

PROOF. Take ϕ, $\psi\in C_0^\infty(\Omega)$ such that $\phi\subset\subset\psi$ and $\phi(x)\equiv 1$ on K. Then, for $u\in C_0^\infty(K)$ we have $\phi\Lambda_{s,k,\varepsilon}u\in C_0^\infty(K')$, where $K'=\operatorname{supp}\phi$. Replacing u by $\phi\Lambda_{s,k,\varepsilon}u$ in (2.2) we obtain

$$\begin{aligned}\|P_{(\beta)}\phi\Lambda_{s,k,\varepsilon}u\|_{-1,\beta|}&\leqq\mu\|P\phi\Lambda_{s,k,\varepsilon}u\|_0+C_1\|\phi\Lambda_{s,k,\varepsilon}u\|_{-N}\\&\leqq\mu\|P\phi\Lambda_{s,k,\varepsilon}u\|_0+C_1'\|u\|_{-N+s},\quad u\in C_0^\infty(K).\end{aligned}$$

We used the estimate (2.17) in the last step. Since N is arbitrary, we have

$$\begin{aligned}&\|P_{(\beta)}\phi\Lambda_{s,k,\varepsilon}u\|_{-1\beta|}\\&\quad\leqq\mu\|P\phi\Lambda_{s,k,\varepsilon}u\|_0+C_1'\|u\|_{-N},\quad u\in C_0^\infty(K).\end{aligned}$$

Noting that $\Lambda_{s,k,\varepsilon}u=\phi\Lambda_{s,k,\varepsilon}u+(1-\phi)\Lambda_{s,k,\varepsilon}\phi u$ for $u\in C_0^\infty(K)$ and $P_{(\beta)}(1-\phi)\cdot\Lambda_{s,k,\varepsilon}\phi\in S^{-\infty}$ from Proposition 2.6, we obtain

$$\|P_{(\beta)}\Lambda_{s,k,\varepsilon}u\|_{-1,\beta|}\leqq\|P_{(\beta)}\phi\Lambda_{s,k,\varepsilon}u\|_{-1,\beta|}+C_2\|u\|_{-N},\quad u\in C_0^\infty(K).$$

In the same way we have

$$\|P\phi\Lambda_{s,k,\varepsilon}u\|_0\leqq\|P\Lambda_{s,k,\varepsilon}u\|_0+C_3\|u\|_{-N},\quad u\in C_0^\infty(K).$$

From the above three estimates it follows that the estimate

$$\|P_{(\beta)}\Lambda_{s,k,\varepsilon}u\|_{-1,\beta|}\leqq\mu\|P\Lambda_{s,k,\varepsilon}u\|_0+C_4\|u\|_{-N},\quad u\in C_0^\infty(K)$$

holds. Using the expansion formula

$$[P, \Lambda_{s,k,\varepsilon}]-\sum_{0<|\alpha|<s+m+N}\frac{(-1)^{|\alpha|}}{\alpha!}\Lambda^{(\alpha)}_{s,k,\varepsilon}P_{(\alpha)}\in S^{-N}$$

in Proposition 2.4, we see that the estimate

$$\begin{aligned}|P\Lambda_{s,k,\varepsilon}u|_0&\leqq\|\Lambda_{s,k,\varepsilon}Pu\|_0+\|[P,\Lambda_{s,k,\varepsilon}]u\|_0\\&\leqq\|\Lambda_{s,k,\varepsilon}Pu\|_0+C_5\Big(\sum_{0<|\alpha|<s+m+N}\|\Lambda^{(\alpha)}_{s,k,\varepsilon}P_{(\alpha)}u\|_0+\|u\|_{-N}\Big)\\&\leqq\|\Lambda_{s,k,\varepsilon}Pu\|_0+C_5'\Big(\sum_{0<|\alpha|<s+m+N}\|\Lambda_{s-|\alpha|,k,\varepsilon}P_{(\alpha)}u\|_0+\|u\|_{-N}\Big),\\&\qquad\qquad u\in C_0^\infty(K)\end{aligned}$$

holds for a constant C_5' (independent of ε). We used the estimate (2.18) in the last step. Similarly we have

$$\begin{aligned}\|\Lambda_{s-|\beta|,k,\varepsilon}P_{(\beta)}u\|_0&\leqq\|P_{(\beta)}\Lambda_{s,k,\varepsilon}u\|_{-|\beta|}\\&+C_6\Big(\sum_{|\beta|<|\alpha|<s+m+N}\|\Lambda_{s-|\alpha|,k,\varepsilon}P_{(\alpha)}u\|_0+\|u\|_{-N}\Big),\quad u\in C_0^\infty(K).\end{aligned}$$

From the above three estimates it follows that for any $0<|\beta|<s+m+N$ the estimate

$$\begin{aligned}\|\Lambda_{s-|\beta|,k,\varepsilon}P_{(\beta)}u\|_0&\leqq\mu\Big(\|\Lambda_{s,k,\varepsilon}Pu\|_0+\sum_{0<|\alpha|<s+m+N}\|\Lambda_{s-|\alpha|,k,\varepsilon}P_{(\alpha)}u\|_0\Big)\\&+C_7\Big(\sum_{|\beta|<|\alpha|<s+m+N}\|\Lambda_{s-|\alpha|,k,\varepsilon}P_{(\alpha)}u\|_0+\|u\|_{-N}\Big),\quad u\in C_0^\infty(K)\end{aligned}$$

holds for a constant C_7 (independent of ε). Applying Lemma 2.9 to the above estimate for $0<|\beta|<s+m+N$, that is, setting $\|\Lambda_{s,k,\varepsilon}Pu\|_0$, $\|\Lambda_{s-|\beta|,k,\varepsilon}P_{(\beta)}u\|_0$ $(0<|\beta|<s+m+N)$ and $\|u\|_{-N}$ for a_0, a_j $(j=1,\cdots,l-1)$ and a_l, respectively, it is easy to see that (2.19) holds. Q. E. D.

COROLLARY. *Let P satisfy Condition* II) *of Definition* 2.1. *Then, for any $K\subset\Omega$, any real s, s', any $N>0$, $\varepsilon>0$ and $k\geqq0$ there exists a constant $C=C(K, s, s', N, k)$ independent of ε such that*

$$\|[P,\Lambda_{s,k,\varepsilon}]u\|_{s'}\leqq C(\|\Lambda_{s+s',k,\varepsilon}Pu\|_0+\|u\|_{-N}),\quad u\in C_0^\infty(K). \tag{2.20}$$

PROOF. From the expansion

$$[P,\Lambda_{s,k,\varepsilon}]-\sum_{0<|\alpha|<s+s'+m+N}C_\alpha\Lambda^{(\alpha)}_{s,k,\varepsilon}P_{(\alpha)}\in S^{-N-s}$$

and (2.19) it is easy to see that (2.20) holds. Q. E. D.

LEMMA 2.11. *Let P satisfy Conditions* I) *and* II) *of Definition* 2.1. *Then, any K, s, $N>0$, $\varepsilon>0$ and $k\geqq 0$ there exists a constant $C=C(K, s, N, k)$ independent of ε such that*

$$\|\Lambda_{s,k,\varepsilon}u\|_0\leqq C(\|\Lambda_{s,k,\varepsilon}Pu\|_0+\|u\|_{-N}),\quad u\in C_0^\infty(K). \tag{2.21}$$

PROOF. Take $\psi\in C_0^\infty(\Omega)$ so that $\psi(x)\equiv 1$ on K, and replace $u\in C_0^\infty(K)$ by $\psi\Lambda_{s,k,\varepsilon}u$ in (2.1). Then, by the same way as in the first half of the proof of Lemma 2.10, we have

$$\|\Lambda_{s,k,\varepsilon}u\|_0\leqq C_1(\|P\Lambda_{s,k,\varepsilon}u\|_0+\|u\|_{-N}),\quad u\in C_0^\infty(K).$$

Using $P\Lambda_{s,k,\varepsilon}=\Lambda_{s,k,\varepsilon}P+[P, \Lambda_{s,k,\varepsilon}]$ and (2.20) we obtain the estimate (2.21). Q. E. D.

REMARK. Set $k=s_0+m+N$ for $s_0>0$. Then, for any $v\in H_{-N}\cap\mathcal{E}'(K)$ the estimate

$$\|\Lambda_{s,k,\varepsilon}v\|_0\leqq C(\|\Lambda_{s,k,\varepsilon}Pv\|_0+\|v\|_{-N}) \tag{2.22}$$

holds, where $s\leqq s_0$ and C is a constant independent of ε. Indeed, taking the sequence $\{u_j\}_{j=1}^\infty$ such that

$$u_j\in C_0^\infty(K')\quad (K\subset K'\subset\Omega),\quad u_j\to v\ \text{ in }\ H_{-N},$$

and noting that $\Lambda_{s,k,\varepsilon}$, $\Lambda_{s,k,\varepsilon}P\in\mathcal{S}^{-N}$ for fixed $\varepsilon>0$ and that the estimate (2.21) holds for $u_j\in C_0^\infty(K')$, we have (2.22) by $j\to\infty$.

LEMMA 2.12. *Let P satisfy Condition* II) *of Definition* 2.1. *Suppose that Q is a differential operator of order m' with coefficients in $\mathcal{B}(R^n)$ and that for any $K\subset\Omega$, real s and $N>0$ there exists a constant $C_1=C_1(K, s, N)$ such that*

$$\|Qu\|_s\leqq C_1(\|Pu\|_{s-\kappa}+\|u\|_{-N}),\quad u\in C_0^\infty(K), \tag{2.23}$$

where κ is some real number. Then, for any β, any $K\subset\Omega$, real s and $N>0$ we can find some constant $C_2=C_2(\beta, K, s, N)$ such that

$$\|Q_{(\beta)}u\|_s\leqq C_2(\|Pu\|_{s-\kappa+|\beta|}+\|u\|_{-N}),\quad u\in C_0^\infty(K). \tag{2.24}$$

Furthermore for any $\varepsilon>0$ and $k\geqq 0$ we can find a constant $C_3=C_3(\beta, K, s, N, k)$ independent of ε such that

$$\|\Lambda_{s-|\beta|,k,\varepsilon}Q_{(\beta)}u\|_0\leqq C_3(\|\Lambda_{s-\kappa,k,\varepsilon}Pu\|_0+\|u\|_{-N}),\quad u\in C_0^\infty(K). \tag{2.25}$$

PROOF. By induction on $|\beta|$ we show that (2.24) holds. When $|\beta|=0$, it is trivial that (2.24) holds. Assume that the estimate (2.24) holds for $|\beta|\leqq n$.

It is easy to see that if $|\beta|=n+1$, then $Q_{(\beta)}=[D_{x_j}, Q_{(\beta')}]$ for $|\beta'|=n$. Using this, for $|\beta|=n+1$ we have

$$\begin{aligned}\|Q_{(\beta)}u\|_s&=\|[D_{x_j}, Q_{(\beta')}]u\|_s\\&\leqq\|D_{x_j}Q_{(\beta')}u\|_s+\|Q_{(\beta')}D_{x_j}u\|_s\\&\leqq C(\|Q_{(\beta')}u\|_{s+1}+\|Q_{(\beta')}D_{x_j}u\|_s),\quad u\in C_0^\infty(K),\end{aligned}$$

where C is some constant (we often denote different constants by same notation C in what follows). By the inductive hypothesis we have

$$\begin{aligned}&\|Q_{(\beta')}u\|_{s+1}+\|Q_{(\beta')}D_{x_j}u\|_s\\&\quad\leqq C(\|Pu\|_{s+1+|\beta'|-\kappa}+\|PD_{x_j}u\|_{s+|\beta'|-\kappa}+\|u\|_{-N}),\\&\qquad\qquad u\in C_0^\infty(K).\end{aligned}$$

From (2.19) of Lemma 2.10 the estimate

$$\begin{aligned}\|PD_{x_j}u\|_s&\leqq\|[P, D_{x_j}]u\|_s+\|D_{x_j}Pu\|_s\\&\leqq C(\|P_{(j)}u\|_s+\|Pu\|_{s+1})\\&\leqq C(\|Pu\|_{s+1}+\|u\|_{-N}),\quad u\in C_0^\infty(K)\end{aligned}$$

holds for any real s. Hence, from the above three estimates we obtain (2.24) with $|\beta|=n+1$. Next we prove (2.25). Replacing u by $\Lambda_{-|\beta|,k,\varepsilon}u$ in (2.24) as in the first half of the proof of Lemma 2.10, we have

$$\begin{aligned}\|Q_{(\beta)}\Lambda_{-|\beta|,k,\varepsilon}u\|_s&\leqq C(\|P\Lambda_{-|\beta|,k,\varepsilon}u\|_{s-\kappa+|\beta|}+\|u\|_{-N})\\&\leqq C(\|\Lambda_{s-\kappa,k,\varepsilon}Pu\|_0+\|[P, \Lambda_{-|\beta|,k,\varepsilon}]u\|_{s-\kappa+|\beta|}+\|u\|_{-N}),\\&\qquad\qquad u\in C_0^\infty(K).\end{aligned}$$

Applying (2.20) in Corollary to Lemma 2.10 to the second term on the right side, we obtain

$$\|Q_{(\beta)}\Lambda_{-|\beta|,k,\varepsilon}u\|_s\leqq C(\|\Lambda_{s-\kappa,k,\varepsilon}Pu\|_0+\|u\|_{-N}),\quad u\in C_0^\infty(K).$$

Using the expansion

$$[Q_{(\beta)}, \Lambda_{-|\beta|,k,\varepsilon}]-\sum_{0<|\alpha|<s+m'+N-|\beta|}\frac{(-1)^{|\alpha|}}{\alpha!}\Lambda^{(\alpha)}_{-|\beta|,k,\varepsilon}Q_{(\alpha+\beta)}\in S^{-N-s},$$

we have

$$\begin{aligned}\|\Lambda_{s-|\beta|,k,\varepsilon}Q_{(\beta)}u\|_0&\leqq\|Q_{(\beta)}\Lambda_{-|\beta|,k,\varepsilon}u\|_s+\|[Q_{(\beta)}, \Lambda_{-|\beta|,k,\varepsilon}]u\|_s\\&\leqq\|Q_{(\beta)}\Lambda_{-|\beta|,k,\varepsilon}u\|_s+C\Big(\sum_{0<|\alpha|<s+m'+N-|\beta|}\|\Lambda^{(\alpha)}_{-|\beta|,k,\varepsilon}Q_{(\alpha+\beta)}u\|_s+\|u\|_{-N}\Big)\end{aligned}$$

$$\leqq \|Q_{(\beta)}\Lambda_{-|\beta|,k,\varepsilon}u\|_s + C\Big(\sum_{|\beta|<|\alpha|\leqq s+m'+N} \|\Lambda_{s-|\alpha|,k,\varepsilon}Q_{(\alpha)}u\|_0 + \|u\|_{-N}\Big),$$

$$u\in C_0^\infty(K).$$

We used (2.18) in the last step. From the above two estimates we obtain for $0\leqq|\beta|<s+m'+N$

$$\|\Lambda_{s-|\beta|,k,\varepsilon}Q_{(\beta)}u\|_0$$

$$\leqq C\Big(\|\Lambda_{s-\kappa,k,\varepsilon}Pu\|_0 + \sum_{|\beta|<|\alpha|\leqq s+m'+N} \|\Lambda_{s-|\alpha|,k,\varepsilon}Q_{(\alpha)}u\|_0 + \|u\|_{-N}\Big),$$

$$u\in C_0^\infty(K).$$

Applying Lemma 2.9 to the above estimate for $0\leqq|\beta|<s+m'+N$, it is easy to see that (2.25) holds. Hence, the lemma is proved. Q.E.D.

REMARK. Set $k=\max(s_0+m+N,\ s_0+m'+N)$ for $s_0>0$, where m and m' are the orders of P and Q, respectively. Then, for any $v\in H_{-N}\cap\mathcal{E}'(K)$ the estimate

$$\|\Lambda_{s,k,\varepsilon}Qv\|_0\leqq C(\|\Lambda_{s-\kappa,k,\varepsilon}Pv\|_0+\|v\|_{-N}) \tag{2.26}$$

holds, where $s\leqq s_0$ and C is a constant independent of ε. This fact is obtained from (2.25) by the same way as in Remark of Lemma 2.10.

LEMMA 2.13. *Let P satisfy Condition* II) *of Definition* 2.1 *and let κ be some real number. Let Q be a differential operator of order m' with coefficients in $\mathcal{B}(R^n)$ such that for any $K\subset\Omega$, any β, some real s_β and any $N>0$ the estimate*

$$\|Q_{(\beta)}u\|_{s_\beta-|\beta|}\leqq C(K,\ \beta,\ N)(\|Pu\|_{s_\beta-\kappa}+\|u\|_{-N}),\quad u\in C_0^\infty(K) \tag{2.27}$$

holds for some constant $C(K,\ \beta,\ N)$. Then, for any real s there exists a constant $C(K,\ \beta,\ N,\ s)$ such that the estimate, replaced s_β by s in (2.27), *holds.*

PROOF. Replacing u by $\Lambda_{s-s_\beta}u$ in (2.27) as in the first half of the proof of Lemma 2.10 (where Λ_{s-s_β} denotes the pseudodifferential operator with symbol $\langle\xi\rangle^{s-s_\beta}$), we have

$$\begin{aligned}\|Q_{(\beta)}\Lambda_{s-s_\beta}u\|_{s_\beta-|\beta|} &\leqq C(\|P\Lambda_{s-s_\beta}u\|_{s_\beta-\kappa}+\|u\|_{-N})\\ &\leqq C(\|Pu\|_{s-\kappa}+\|[P,\ \Lambda_{s-s_\beta}]u\|_{s_\beta-\kappa}+\|u\|_{-N})\\ &\leqq C(\|Pu\|_{s-\kappa}+\|u\|_{-N}),\quad u\in C_0^\infty(K).\end{aligned}$$

Here we used (2.20) with $k=0$ in the last step. Furthermore, we have

$$\|Q_{(\gamma,\delta)}u\|_{s-1,\delta|} \leqq \|Q_{(\gamma,\delta)}\Lambda_{s-s_{\gamma\delta},\delta}u\|_{s_{\gamma\delta}-1,\delta|} + \|[Q_{(\gamma,\delta)}, \Lambda_{s-s_{\gamma\delta}}]u\|_{s_{\gamma\delta}-1,\delta|}$$

$$\leqq C\Big(\|Pu\|_{s-\kappa} + \sum_{1,\delta<|\alpha|<s+m'+N} \|Q_{(\alpha)}u\|_{s-|\alpha|} + \|u\|_{-N}\Big), \qquad u\in C_0^\infty(K).$$

Therefore, by Lemma 2.9 we obtain

$$\|Q_{(\gamma,\delta)}u\|_{s-,\delta|} \leqq C(\|Pu\|_{s-\kappa} + \|u\|_{-N}), \quad u\in C_0^\infty(K).$$

Q. E. D.

We end this section by completing the proof of Theorem 2.2, that is, we prove that (2.4) holds.

THE PROOF OF THEOREM 2.2. Let x_0 be any fixed point in Ω' and let $\psi(x)\in C_0^\infty(\Omega')$ such that $\psi(x)\equiv 1$ in a neighborhood $U(x_0)$ of x_0. Then, for any natural number l we can find a finite sequence $\{\psi_j(x)\}_{j=1}^l \subset C_0^\infty(\Omega')$ such that

$$\psi_1 \subset\subset \psi_2 \subset\subset \cdots \subset\subset \psi_l \subset\subset \psi,$$
$$\psi_1(x)\equiv 1 \quad \text{in a neighborhood of} \quad x_0,$$

and we have

$$\|[P, \psi_j]u\|_s \leqq C(K, s, N, \psi_j)(\|Pu\|_{s-\kappa} + \|u\|_{-N}), \tag{2.28}$$
$$(j=1, \cdots, l), \quad u\in C_0^\infty(K),$$

for any $K\subset\Omega$, real s and $N>0$, where κ is some positive number. Indeed, from Condition III) of Definition 2.1 we can take $\psi_1(x)\in C_0^\infty(U(x_0))$ such that $\psi_1(x)\equiv 1$ in some neighborhood $V(x_0)$ of x_0 and satisfies (2.3). It is clear that $\psi_1 \subset\subset \psi$. For x_0 and $V(x_0)$ we can take again $\psi_2(x)\in C_0^\infty(V(x_0))$ such that $\psi_2(x)\equiv 1$ in some neighborhood of x_0 and satisfies (2.3). Repeating these steps l times, we have $\{\psi_j(x)\}_{j=1}^l \subset C_0^\infty(\Omega')$. Set $\psi_j=\psi_{l-j+1}$ $(j=1, \cdots, l)$. Then, $\{\psi_j\}_{j=1}^l$ is a desired sequence. As well-known, for $\psi v\in\mathcal{E}'$ there exists $N>0$ such that $\psi v\in H_{-N}$. Let us choose l bigger than $(s+m+N)/\kappa$. By Remark of Lemma 2.11, for $\psi_j v\in H_{-N}\cap\mathcal{E}'(K)$ (, where $K=\operatorname{supp}\psi$) the estimate

$$\|\Lambda_{s,k,\varepsilon}\psi_1 v\|_0 \leqq C(\|\Lambda_{s,k,\varepsilon}P\psi_1 v\|_0 + \|\psi_1 v\|_{-N}) \tag{2.29}$$

holds for C independent of ε and $k=s+m+N$. (Let the same notation C denote different constants independent of ε in what follows.) Furthermore, setting $Q=[P, \psi_j]$ in Lemma 2.12, from (2.28) and Remark of Lemma 2.12 it is easy to see that if $k=s+m+N$, then for $s'\leqq s$ the estimate

(2.30) $$\|\Lambda_{s',k,\varepsilon}[P,\psi_j]\psi_{j+1}v\|_0 \leqq C(\|\Lambda_{s'-\kappa,k,\varepsilon}P\psi_{j+1}v\|_0+\|\psi_{j+1}v\|_{-N})$$

holds. Noting $[P,\psi_j]\psi_{j+1}=P\psi_j-\psi_jP$, we have from (2.29)

$$\|\Lambda_{s,k,\varepsilon}\psi_1 v\|_0 \leqq C(\|\Lambda_{s,k,\varepsilon}\psi_1 Pv\|_0+\|\Lambda_{s,k,\varepsilon}[P,\psi_1]\psi_2 v\|_0+\|\psi_1 v\|_{-N}).$$

Applying (2.30) to the second term on the right side, and noting

$$\|\psi_j v\|_{-N}=\|\psi_j\psi v\|_{-N}\leqq C\|\psi v\|_{-N},$$

we obtain

$$\begin{aligned}\|\Lambda_{s,k,\varepsilon}\psi_1 v\|_0 &\leqq C(\|\Lambda_{s,k,\varepsilon}\psi_1 Pv\|_0+\|\Lambda_{s-\kappa,k,\varepsilon}P\psi_2 v\|_0+\|\psi v\|_{-N})\\ &\leqq C(\|\Lambda_{s,k,\varepsilon}\psi_1 Pv\|_0+\|\Lambda_{s-\kappa,k,\varepsilon}\psi_2 Pv\|_0\\ &\quad+\|\Lambda_{s-\kappa,k,\varepsilon}[P,\psi_2]\psi_3 v\|_0+\|\psi v\|_{-N}).\end{aligned}$$

Applying (2.30) to third term on the right side, and repeating the same procedure, we have

$$\|\Lambda_{s,k,\varepsilon}\psi_1 v\|_0 \leqq C\Big(\sum_{j=1}^{l}\|\Lambda_{s-\kappa(j-1),k,\varepsilon}\psi_j Pv\|_0+\|\Lambda_{s-\kappa l,k,\varepsilon}P\psi v\|_0+\|\psi v\|_{-N}\Big).$$

Since $\psi_j Pv\in H_s$ from the hypothesis of theorem, and since $\Lambda_{s-\kappa l,k,\varepsilon}P\in S^{-N}$ for any ε, we obtain from (2.17)

$$\begin{aligned}\|\Lambda_{s,k,\varepsilon}\psi_1 v\|_0 &\leqq C\Big(\sum_{j-1}^{l}\|\psi_j Pv\|_{s-\kappa(j-1)}+\|\psi v\|_{-N}\Big)\\ &\leqq C(\|\psi Pv\|_s+\|\psi v\|_{-N}).\end{aligned}$$

Letting $\varepsilon\to+0$, we finally get (2.4) with $\phi=\psi_1$. The theorem is proved.

Q. E. D.

§ 3. Proof of Theorem 1.1 (Part I).

Let $P=p(x,y,D_x,D_y)=A+gB=a(x,y,D_x)+g(x)b(x,y,D_y)$ denote a differential operator in Theorem 1.1. We shall show that P is w-elliptic in R^n. Since the conditions of Definition 2.1 are stated for some compact set K of R^n, we may assume, without loss of generality, that $g(x)$ and coefficients of A and B belong to $\mathcal{B}$, and $g(x)$ satisfies for any $\varepsilon>0$

(3.1) $$g(x)\geqq C_\varepsilon>0 \quad \text{on} \quad \{|x|\geqq\varepsilon\}.$$

LEMMA 3.1. *Set* $\Omega_\varepsilon=\{(x,y)\in R^n;\ |x|<\varepsilon\}$. *Then, for any* $\varepsilon>0$, *any* α, *any real* s *and any* $N>0$ *there exist constants* $C(\varepsilon,s,N)$ *and* $C(\varepsilon,\alpha,s,N)$ *such that*

$$\|u\|_s \leqq C(\varepsilon, s, N)(\|Pu\|_{s-\kappa_0}+\|u\|_{-N}), \tag{3.2}$$

$$\|P^{(\alpha)}u\|_s \leqq C(\varepsilon, \alpha, s, N)(\|Pu\|_{s-\rho_0|\alpha|}+\|u\|_{-N}), \quad u\in C_0^\infty(R^n-\Omega_\varepsilon), \tag{3.3}$$

where $\kappa_0=\min(2l, 2m)$ *and* $\rho_0=\min(m/l, l/m)$, *respectively.*

PROOF. Modifying $g(x)$ in $\{|x|<\varepsilon\}$, we may assume that (3.1) holds for any $x\in R^{n_1}_x$. As far as we consider the estimates for $u\in C_0^\infty(R^n-\Omega_\varepsilon)$, such modification is permitted. Set $k=\max(2l, 2m)$ and set $\lambda(\xi, \eta)=(|\xi|^{2l}+|\eta|^{2m}+1)^{1/k}$. Then $\lambda=\lambda(\xi, \eta)$ is a basic weight function and $p(x, y, \xi, \eta)$ belongs to $S^k_{\lambda,1,0}$. From (1.1), (1.2) and (3.1) we obtain

$$|p(x, y, \xi, \eta)| \geqq C_\varepsilon \lambda^k(\xi, \eta) \quad \text{for large} \quad |\xi|+|\eta|,$$

where $C_\varepsilon>0$ depends on ε. By Proposition 2.7 and its Remark we have a parametrix $Q\in S^{-k}_{\lambda,1,0}$ such that $I=QP+K$, $K\in S^{-\infty}$. It is easy to check that $Q\in S^{-\kappa_0}_{\rho_0,0}$ and $P^{(\alpha)}Q\in S^{-\rho_0|\alpha|}_{\rho_0,0}$. Therefore, noting that the semi-norms of $\sigma(Q)$ depend on ε, by means of Proposition 2.5 we obtain (3.2) and (3.3).

Q. E. D.

LEMMA 3.2. *Let* $\phi(x)\in C^\infty(R^{n_1}_x)$ *such that for any* $\alpha\neq 0$, $\phi_{(\alpha)}=0$ *on* $\{|x|\leqq\varepsilon\}$, *where* $\phi_{(\alpha)}=D^\alpha_x\phi(x)$. *Then, for any* $\varepsilon>0$, *any real* s *and any* $N>0$ *there exists a constant* $C(\varepsilon, s, N)$ *such that*

$$\|[P, \phi]u\|_s \leqq C(\varepsilon, s, N)(\|Pu\|_{s-\rho_0}+\|u\|_{-N}), \quad u\in C_0^\infty(R^n), \tag{3.4}$$

where $\rho_0=\min(l/m, m/l)$.

PROOF. Let $u\in C_0^\infty(R^n)$. Then, $\phi_{(\alpha)}u\in C_0^\infty(R^n-\Omega_\varepsilon)$ for any $\alpha\neq 0$. Hence, by Lemma 3.1 we have for any $\alpha\neq 0$ and $\beta\neq 0$

$$\|P^{(\beta)}\phi_{(\alpha)}u\|_s \leqq C(\|P\phi_{(\alpha)}u\|_{s-\rho_0}+\|u\|_{-N}), \quad u\in C_0^\infty(R^n). \tag{3.5}$$

Using this we obtain

$$\begin{aligned}
\|[P, \phi]u\|_s &\leqq C\sum_{1\leqq|\alpha|\leqq 2l}\|P^{(\alpha)}\phi_{(\alpha)}u\|_s \\
&\leqq C\Big(\sum_{1\leqq|\alpha|\leqq 2l}\|P\phi_{(\alpha)}u\|_{s-\rho_0}+\|u\|_{-N}\Big) \\
&\leqq C\Big(\|Pu\|_{s-\rho_0}+\sum_{1\leqq|\alpha|\leqq 2l}\|[P, \phi_{(\alpha)}]u\|_{s-\rho_0}+\|u\|_{-N}\Big) \\
&\leqq C\Big(\|Pu\|_{s-\rho_0}+\sum_{\substack{1\leqq|\alpha|\leqq 2l\\ 1\leqq|\beta|\leqq 2l}}\|P^{(\beta)}\phi_{(\alpha+\beta)}u\|_{s-\rho_0}+\|u\|_{-N}\Big), \\
&\qquad u\in C_0^\infty(R^n).
\end{aligned}$$

Applying (3.5) to the second term on the right side, we have

$$\|[P, \phi]u\|_s \leqq C(\|Pu\|_{s-\rho_0} + \sum_{2\leqq|\alpha|\leqq 4l} \|P\phi_{(\alpha)}u\|_{s-2\rho_0} + \|u\|_{-N}),$$

$$u \in C_0^\infty(R^n).$$

Repeating this procedure j times, we obtain

$$\|[P, \phi]u\|_s \leqq C(\|Pu\|_{s-\rho_0} + \sum_{j\leqq|\alpha|\leqq 2jl} \|P\phi_{(\alpha)}u\|_{s-j\rho_0} + \|u\|_{-N}),$$

$$u \in C_0^\infty(R^n).$$

If we choose j so that $s-\rho_0 j+2l<-N$, we obtain (3.4). Q. E. D.

LEMMA 3.3. *For any $\mu>0$ and any $N>0$ there exists a constant $C(\mu, N)$ such that*

$$\|u\|_0 \leqq \mu\|Pu\|_0 + C(\mu, N)\|u\|_{-N}, \quad u \in C_0^\infty(R^n). \tag{3.6}$$

PROOF. From Conditions 1°), 2°) and 3°) for P it is easy to see that

$$\begin{aligned} \|\langle D_x\rangle^l u\|_0^2 &\leqq C(\mathrm{Re}(Pu, u) + \|u\|_0^2) \\ &\leqq C(\|Pu\|_0^2 + \|u\|_0^2), \quad u \in C_0^\infty(R^n). \end{aligned} \tag{3.7}$$

Here $\langle D_x\rangle^l$ is an operator defined by

$$\int e^{i\langle x\cdot\xi+y\cdot\eta\rangle}\langle\xi\rangle^l \hat{u}(\xi, \eta)d\xi\, d\eta, \quad u \in \mathcal{S}.$$

Let $\phi(x) \in C_0^\infty(R^n)$ such that $\mathrm{supp}\,\phi \subset \{|x|<\varepsilon\}$, $\phi(x)\equiv 1$ on $\{|x|\leqq\varepsilon/2\}$. Then, on account of Poincaré's inequality we have

$$\|\phi u\|_0 \leqq \delta(\varepsilon)\|\langle D_x\rangle^l \phi u\|_0, \quad u \in C_0^\infty(R^n),$$

where $\delta(\varepsilon) \to 0$ $(\varepsilon \to 0)$. From this and the estimate obtained by setting $u=\phi u$ in (3.7) we have

$$\|\phi u\|_0 \leqq C\delta(\varepsilon)(\|P\phi u\|_0 + \|\phi u\|_0), \quad u \in C_0^\infty(R^n).$$

Hence, taking a sufficiently small $\varepsilon>0$ for any fixed $\mu>0$, we obtain

$$\|\phi u\|_0 \leqq \mu\|P\phi u\|_0, \quad u \in C_0^\infty(R^n).$$

Since $(1-\phi)u \in C_0^\infty(R^n-\Omega_\varepsilon)$, from (3.2) we get

$$\|(1-\phi)u\|_0 \leqq C_\varepsilon(\|P(1-\phi)u\|_{-\kappa_0} + \|u\|_{-N}), \quad u \in C_0^\infty(R^n)$$

for a constant C_ε depending on ε. From the above two estimates we have

$$\|u\|_0 \leqq \mu\|P\phi u\|_0 + C_\mu(\|P(1-\phi)u\|_{-\kappa_0} + \|u\|_{-N})$$
$$\leqq \mu C\|Pu\|_0 + C'_\mu(\|Pu\|_{-\kappa_0} + \|[P, \phi]u\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(R^n).$$

Since $\phi(x)$ satisfies the hypothesis of Lemma 3.2, by applying the estimate (3.4) to the third term on the right side we obtain

$$\|u\|_0 \leqq \mu C\|Pu\|_0 + C''_\mu(\|Pu\|_{-\rho_0} + \|u\|_{-N}), \quad u \in C_0^\infty(R^n).$$

Using the interpolation inequality to the second term on the right side we finally obtain (3.6). Thus, the lemma is proved. Q. E. D.

By Lemma 3.3, immediately we see that P satisfies Condition I) of Definition 2.1.

§ 4. Proof of Theorem 1.1 (Part II).

To show that P satisfies Conditions II) and III) of Definition 2.1, we prepare

LEMMA 4.1. *For some $0<\kappa<1$ there exists a constant C_0 such that*

$$\|g(x)\langle D_y\rangle^{2m-1+\kappa}u\|_0 \leqq C_0(\|Pu\|_0 + \|u\|_0), \quad u \in C_0^\infty(R^n). \tag{4.1}$$

Furthermore, for any β $(0<|\beta|<2m)$ and any $\mu>0$ there exists a constant $C(\mu, \beta)$ such that

$$\|g_{(\beta)}(x)\langle D_y\rangle^{2m}u\|_{-1,\beta|} \leqq \mu\|Pu\|_0 + C(\mu, \beta)\|u\|_0, \quad (|\beta|\neq 0) \quad u \in C_0^\infty(R^n). \tag{4.2}$$

Here $\langle D_y\rangle^s u$ is an operator defined by

$$\int e^{i(x\cdot\xi+y\cdot\eta)}\langle\eta\rangle^s \hat{u}(\xi, \eta)\, d\xi\, d\eta \qquad \text{for} \quad u \in \mathcal{S}.$$

Since the proof of this lemma is so long, it will be given in the next section.

Using Lemmas 3.3 and 4.1 we obtain the following two lemmas.

LEMMA 4.2. *For any $|\beta|\neq 0$, any $\mu>0$ and any $N>0$ there exists a constant $C=C(\beta, \mu, N)$ such that*

$$\|P_{(\beta)}u\|_{-1,\beta|} \leqq \mu\|Pu\|_0 + C\|u\|_{-N}, \quad (|\beta|\neq 0) \quad u \in C_0^\infty(R^n). \tag{4.3}$$

PROOF. We note

$$P_{(\beta)} = A_{(\beta)} + \sum_{\beta'+\beta''=\beta} C_{\beta',\beta''} g_{(\beta')} B_{(\beta'')}.$$

First we observe that

$$\|A_{(\beta)}u\|_{-|\beta|}\leqq C_\beta(\|Au\|_{-1}+\|u\|_0),\quad (|\beta|\neq 0)\quad u\in C_0^\infty(R^n). \tag{4.4}$$

If $|\beta|\geqq 2l$, then (4.4) is obvious. For $|\beta|<2l$, we have

$$\begin{aligned}\|A_{(\beta)}u\|_{-|\beta|}&=\|\Lambda_{-|\beta|}A_{(\beta)}u\|_0\\ &\leqq C(\|A_{(\beta)}\Lambda_{-|\beta|}u\|_0+\sum_{0<|\alpha|\leqq 2l-|\beta|}\|\Lambda_{-|\beta+\alpha|}A_{(\beta+\alpha)}u\|_0+\|u\|_0),\\ &\qquad u\in C_0^\infty(R^n),\end{aligned} \tag{4.5}$$

where Λ_s is a pseudodifferential operator with the symbol $(|\xi|^2+|\eta|^2+1)^{s/2}$. From Condition 1°) for A it follows that

$$\|A^*_{(\beta)}u\|_0\leqq C(\|Au\|_0+\|u\|_0),\quad u\in C_0^\infty(R^n).$$

Applying this estimate to the first term on the right side of (4.5), we obtain

$$\begin{aligned}\|A_{(\beta)}u\|_{-|\beta|}&\leqq C(\|A\Lambda_{-|\beta|}u\|_0+\sum_{0<|\alpha|\leqq 2l-|\beta|}\|\Lambda_{-|\beta+\alpha|}A_{(\beta+\alpha)}u\|_0+\|u\|_0)\\ &\leqq C(\|Au\|_{-|\beta|}+\sum_{0<|\alpha|\leqq 2l-|\beta|}\|\Lambda_{-|\beta+\alpha|}A_{(\alpha)}u\|_0\\ &\quad+\sum_{0<|\alpha|\leqq 2l-|\beta|}\|A_{(\beta+\alpha)}u\|_{-|\beta+\alpha|}+\|u\|_0),\quad u\in C_0^\infty(R^n).\end{aligned}$$

Using the interpolation inequality, we have for $\mu>0$ and $0<|\beta|<2l$

$$\begin{aligned}\|A_{(\beta)}u\|_{-|\beta|}&\leqq C(\|Au\|_{-1}+\sum_{|\beta|<|\alpha|<l}\|A_{(\alpha)}u\|_{-|\alpha|}+\|u\|_0)\\ &\quad+\mu\sum_{0<|\alpha|<2l}\|A_{(\alpha)}u\|_{-|\alpha|},\quad u\in C_0^\infty(R^n).\end{aligned}$$

By means of Lemma 2.9, we obtain (4.4). On the other hand

$$\begin{aligned}\|Au\|_{-1}&\leqq\|Pu\|_{-1}+\|g(x)Bu\|_{-1}\\ &\leqq\mu\|Pu\|_0+C(\|g\langle D_y\rangle^{2m-1}u\|_0+\|u\|_0),\quad u\in C_0^\infty(R^n).\end{aligned}$$

Combining this with (4.4), we have

$$\|A_{(\beta)}u\|_{-|\beta|}\leqq\mu|Pu|_0+C(\|g\langle D_y\rangle^{2m-1}u\|_0+\|u\|_0),\quad (|\beta|\neq 0)\quad u\subset C_0^\infty(R^n).$$

Noting that

$$\|g(x)\langle D_y\rangle^{2m-1}u\|_0\leqq\mu\|g(x)\langle D_y\rangle^{2m-1+\kappa}u\|_0+C_\mu\|u|_0,\quad u\in C_0^\infty(R^n),$$

we obtain from (4.1)

$$\|A_{(\beta)}u\|_{-|\beta|}\leqq\mu\|Pu\|_0+C\|u\|_0,\quad (|\beta|\neq 0)\quad u\in C_0^\infty(R^n). \tag{4.6}$$

Using (4.1) and (4.2) we get for $0<|\beta|<2m$

$$\sum_{\beta'+\beta''=\beta}\|g_{(\beta')}B_{(\beta'')}u\|_{-|\beta|}\leqq C\sum_{|\beta'|\leqq|\beta|}\|g_{(\beta')}\langle D_y\rangle^{2m}u\|_{-|\beta|} \tag{4.7}$$

$$\leqq C\Big(\sum_{0<|\beta'|<2m}\|g_{(\beta')}\langle D_y\rangle^{2m}u\|_{-|\beta'|}+\|g\langle D_y\rangle^{2m-1}u\|_0\Big)$$

$$\leqq\mu\|Pu\|_0+C\|u\|_0, \qquad u\in C_0^\infty(R^n).$$

If $|\beta|\geqq 2m$, then the left side of (4.7) is bounded only by $C\|u\|_0$. Therefore, from (4.6) and (4.7) we have

$$\|P_{(\beta)}u\|_{-|\beta|}\leqq\mu\|Pu\|_0+C\|u\|_0, \quad (|\beta|\neq 0) \quad u\in C_0^\infty(R^n).$$

By means of (3.6) we obtain (4.3). Q.E.D.

LEMMA 4.3. *Let $\phi(x, y)=\phi_1(x)\phi_2(y)$ $(\in C_0^\infty(R^n))$ where $\phi_1\equiv 1$ on $\{|x|<\varepsilon\}$, $\phi_1(x)\in C_0^\infty(R_x^{n_1})$, $\phi_2(y)\in C_0^\infty(R_y^{n_2})$. Then, for any $\varepsilon>0$, real s and $N>0$ there exists a constant $C=C(\varepsilon, s, N)$ such that*

$$\|[P, \phi]u\|_s\leqq C(\|Pu\|_{s-\kappa'}+\|u\|_{-N}), \quad u\in C_0^\infty(R^n), \tag{4.8}$$

where $\kappa'=\min(l\kappa/(2m-1+\kappa), \kappa)$ and κ is the positive number in Lemma 4.1.

PROOF. We devide the proof into three steps.

i) Assume that for any $\alpha\neq 0$, β, real s and $N>0$ there exists a constant C such that

$$\|P^{(\alpha)}\phi_{(\alpha+\beta)}u\|_s\leqq C(\|P\phi_{(\alpha+\beta)}u\|_{s-\kappa'}+\|u\|_{-N}), \quad u\in C_0^\infty(R^n). \tag{4.9}$$

Then, (4.8) is valid. Indeed, using (4.9) we have

$$\|[P, \phi]u\|_s\leqq C\sum_{|\alpha|>0}\|P^{(\alpha)}\phi_{(\alpha)}u\|_s$$

$$\leqq C\Big(\sum_{|\alpha|>0}\|P\phi_{(\alpha)}u\|_{s-\kappa'}+\|u\|_{-N}\Big)$$

$$\leqq C\Big(\|Pu\|_{s-\kappa'}+\sum_{|\alpha|\neq 0}\|[P, \phi_{(\alpha)}]u\|_{s-\kappa'}+\|u\|_{-N}\Big)$$

$$\leqq C\Big(\|Pu\|_{s-\kappa'}+\sum_{|\alpha|\neq 0}\sum_{|\beta|\neq 0}\|P^{(\beta)}\phi_{(\alpha+\beta)}u\|_{s-\kappa'}+\|u\|_{-N}\Big),$$

$$u\in C_0^\infty(R^n).$$

Applying (4.9) to the second term on the right side, again, we get

$$\|[P, \psi]u\|_s \leqq C\Big(\|Pu\|_{s-\kappa'} + \sum_{|\alpha|\neq 0} \sum_{|\beta|\neq 0} \|P\psi_{(\alpha+\beta)}u\|_{s-2\kappa'} + \|u\|_{-N}\Big)$$

$$\leqq C\Big(\|Pu\|_{s-\kappa'} + \sum_{2\leqq|\alpha|\leqq 2k} \|[P, \psi_{(\alpha)}]u\|_{s-2\kappa'} + \|u\|_{-N}\Big),$$

$$u\in C_0^\infty(R^n),$$

where $k=\max(2l, 2m)$. Repeating this procedure, as in the proof of Lemma 3.2, we finally obtain (4.8).

ii) We prove (4.9). Let $\alpha=(\alpha_1, \alpha_2)$ where α_1 and α_2 are multi-indices with respect to x and y, respectively. If $\alpha_1\neq 0$, then for any β, $\psi_{(\alpha+\beta)}=0$ on $\Omega_\varepsilon=\{(x, y); |x|\leqq\varepsilon\}$. Hence, by Lemma 3.1, the estimate (4.9) clearly holds. When $\alpha_1=0$, that is, $\alpha=(0, \alpha_2)$, $\alpha_2\neq 0$, we prove

$$\|P^{(\alpha)}u\|_s \leqq C(\|Pu\|_{s-\kappa'} + \|u\|_{-N}), \quad u\in C_0^\infty(R^n). \tag{4.10}$$

Then, replacing u by $\psi_{(\alpha+\beta)}u$, we get easily (4.9). Since we have

$$\|P^{(\alpha)}u\|_s = \|B^{(\alpha_2)}(x, y, D_y)\langle D_y\rangle^{1-2m} g(x)\langle D_y\rangle^{2m-1}u\|_s$$
$$\leqq C\|g(x)\langle D_y\rangle^{2m-1}u\|_s, \quad u\in C_0^\infty(R^n),$$

the estimate (4.10) is obtained if we derive the estimate

$$\|g\langle D_y\rangle^{2m-1}u\|_s \leqq C(\|Pu\|_{s-\kappa'} + \|u\|_{-N}), \quad u\in C_0^\infty(R^n). \tag{4.11}$$

iii) On account of Lemma 2.13, the estimate (4.11) holds if we prove that the estimate

$$\|g_{(\beta)}\langle D_y\rangle^{2m-1}u\|_{\kappa'-|\beta|} \leqq C(\|Pu\|_0 + \|u\|_{-N}), \quad u\in C_0^\infty(R^n) \tag{4.12}$$

holds for any β and $N>0$. (Though $g(x)\langle D_y\rangle^{2m-1}$ is not a differential operator, we can apply Lemma 2.13 to this operator since $\|g(x)\langle D_y\rangle^{2m-1}u\|_{\kappa'-|\beta|}$ is equivalent to $\sum_{|\alpha|\leqq 2m-1} \|g(x)D_y^\alpha u\|_{\kappa'-|\beta|}$.) By means of Minkovski's inequality, it is easy to see

$$(|\xi|+|\eta|+1)^{\kappa'}\langle\eta\rangle^{2m-1} \leqq \langle\xi\rangle^{\kappa'}\langle\eta\rangle^{2m-1} + \langle\eta\rangle^{2m-1+\kappa'}$$
$$\leqq C(\langle\xi\rangle^l + \langle\eta\rangle^{2m-1+\kappa'(2m-1)/(l-\kappa')} + \langle\eta\rangle^{2m+\kappa'-1}).$$

If we take κ' so that $\max(\kappa'(2m-1)/(l-\kappa'), \kappa')\leqq\kappa$, where κ is the positive number in Lemma 4.1, we have for any β

$$\|g_{(\beta)}\langle D_y\rangle^{2m-1}u\|_{\kappa'-|\beta|}$$
$$\leqq C(\|\langle D_x\rangle^l g_{(\beta)}u\|_{-|\beta|} + \|g_{(\beta)}\langle D_y\rangle^{2m-1+\kappa}u\|_{-|\beta|}$$
$$\leqq C(\|\langle D_x\rangle^l u\|_0 + \|g_{(\beta)}\langle D_y\rangle^{2m-1+\kappa}u\|_{-|\beta|}, \quad u\in C_0^\infty(R^n).$$

Using (3.7), the first term on the right side is bounded by $C(\|Pu\|_0+\|u\|_0)$. By Lemma 4.1 the second term is also bounded by $C(\|Pu\|_0+\|u\|_0)$ since $\kappa<1$. Therefore, using (3.6) we have (4.12). Q.E.D.

The completion of the proof of Theorem 1.1. By Lemmas 3.3 and 4.2 we see that P satisfies I) and II) of Definition 2.1, respectively. Lemma 4.3 shows that Condition III) is satisfied for points $(0, y)$ $(y\in R^{n_2})$. For points (x, y) with $x\neq 0$, take $\psi(x, y)\in C_0^\infty(R^n)$ such that $\operatorname{supp}\psi\subset\{|x|\geqq\varepsilon\}$. Then, we can get the estimate (4.8) with $\kappa'=\min(l/m, m/l)$, by the same way as in the proof of Lemma 3.2. Therefore Condition III) is satisfied for points (x, y) with $x\neq 0$. Consequently, P is w-elliptic in R^n. Q.E.D.

§ 5. Proof of Lemma 4.1.

From Condition (G) of Section 1 or Remark of Theorem 1.1 we have for any β

$$|g_{(\beta)}(x)|\leqq Cg(x)^{1-\sigma|\beta|} \quad \text{in a neighborhood of } x=0, \tag{5.1}$$

where

$$\begin{cases} \sigma=1/2 & \text{when } m=1 \\ 0<\sigma<\{2(m+l(m-1))\}^{-1} & \text{when } m\geqq 2. \end{cases} \tag{5.2}$$

Since P is semi-elliptic in $R^n-\{x=0\}$, it is easy to see that (4.1) and (4.2) hold for $u\in C_0^\infty(R^n-\{x=0\})$, by the same way as in the proof of Lemma 3.1. We may only prove Lemma 4.1 in a neighborhood of $x=0$. Therefore we may assume, without loss of generality, that (5.1) holds for all x, by modifying $g(x)$ outside of $x=0$.

Let $\phi_0(t)$, $\phi_1(t)$ and $\phi_2(t)$ be C^∞-functions in $[0, \infty)$ such that

$$\begin{aligned} &\operatorname{supp}\phi_0(t)\subset[0, 1), \quad \phi_0(t)\equiv 1 \quad \text{on } [0, 1/2], \\ &\operatorname{supp}\phi_1(t)\subset[0, 2), \quad \phi_1(t)\equiv 1 \quad \text{on } [0, 1], \\ &\operatorname{supp}\phi_2(t)\subset(1, \infty), \quad \phi_2(t)\equiv 1 \quad \text{in } [2, \infty), \end{aligned}$$

and

$$\phi_1+\phi_2\equiv 1 \quad \text{in } [0, \infty). \tag{5.3}$$

Set $\lambda(\xi, \eta)=(|\xi|^{2l(l+1)(l+2)}+\langle\eta\rangle^{2m\tau(l+1)(l+2)})^{1/2l(l+1)(l+2)}$

$$\begin{cases} \tau=(l+1)/(l+2) & \text{for } m=1 \\ \tau=l/(m(l+1)) & \text{for } m\geqq 2. \end{cases} \tag{5.4}$$

Then, since $2m\tau(l+1)(l+2)$ is an integer, λ satisfies (2.5) and (2.6), so it is a basic weight function.

PROPOSITION 5.1. *Set* $h(\xi,\eta)=\lambda(\xi,\eta)^{2l(1-\tau)/\tau}$ *and set* $\chi_j(x,\xi,\eta)=\phi_j(g(x)h(\xi,\eta))$ $(j=0, 1, 2)$. *Then,* $\chi_j(x, D_x, D_y)\in S^0_{\lambda,\mathbf{1},\delta}$, *where* $\mathbf{1}=(1, \cdots, 1)$ *and* $\delta=(\delta_1, \cdots, \delta_{n_1}, 0, \cdots, 0)$, $\delta_k=\delta_0=2l\sigma(1-\tau)/\tau$ $(k=1, \cdots, n_1)$. *Furthermore*

$$\chi_1+\chi_2=I. \tag{5.5}$$

PROOF. The equality (5.5) is obvious from (5.3). We check that $\chi(x,\xi,\eta)$ $(=\chi_j(x,\xi,\eta), j=0, 1, 2)$ satisfies for any α and β

$$|(\partial_\xi\partial_\eta)^\alpha D_x^\beta \chi(x,\xi,\eta)|\leqq C_{\alpha,\beta}\lambda^{\delta\cdot\beta-|\alpha|} \quad \text{in} \quad R^n_{x,y}\times R^n_{\xi,\eta}. \tag{5.6}$$

From the Leibnitz formula we have for $|\alpha+\beta|\neq 0$

$$\chi^{(\alpha)}_{(\beta)}=\sum_{0<k\leqq|\alpha+\beta|} C_k\phi^{(k)}(g(x)h(\xi,\eta)) \times \sum^{*}_{\substack{\alpha^1+\alpha^2+\cdots+\alpha^k=\alpha\\ \beta^1+\beta^2+\cdots+\beta^k=\beta}} C_{\alpha^1,\cdots,\alpha^k,\beta^1,\cdots,\beta^k} g^{(\beta^1)}\cdots g^{(\beta^k)} h^{(\alpha^1)}\cdots h^{(\alpha^k)}.$$

Using (5.1) and $|h^{(\tilde{\alpha})}|\leqq C_{\tilde{\alpha}}h\lambda^{-|\tilde{\alpha}|}$, we obtain

$$|\chi^{(\alpha)}_{(\beta)}|\leqq C\sum_{0<k\leqq|\alpha+\beta|}\phi^{(k)}(g(x)h(\xi,\eta))\,g(x)^{k-\sigma|\beta|}h^k\lambda^{-|\alpha|}.$$

Noting that for $k\neq 0$,

$$1/2\leqq g(x)h(\xi,\eta)\leqq 2 \quad \text{on} \quad \operatorname{supp}\phi^{(k)}(g(x)h(\xi,\eta)),$$

we obtain (5.6). Q. E. D.

REMARK. We have proved that $\chi_j\in S^0_{\lambda,\mathbf{1},\delta}$. Recall that by Definition 2.3 $\delta_0<1$, that is

$$2l\sigma(1-\tau)/\tau<1. \tag{5.7}$$

On account of (5.2) and (5.4) it is easy to see that (5.7) is valid.

We shall first see that the estimates with replaced u by $\chi_j u$ $(j=1, 2)$ in (4.1) and (4.2), hold, and prove that the estimate

$$\|[P,\chi_j]u\|_0\leqq C(\|Pu\|_0+\|u\|_0), \quad u\in C_0^\infty(R^n) \tag{5.8}$$

holds.

PROPOSITION 5.2. *Set* $v_1=\chi_1(x, D_x, D_y)u$ *for* $u\in C_0^\infty(R^n)$. *Then, we can find a constant* C_0 *such that*

$$\|g(x)\langle D_y\rangle^{2m-1+\kappa}v_1\|_0\leqq C_0(\|Pv_1\|_0+\|v_1\|_0), \tag{5.9}$$

where $\kappa=2\sigma m(1-\tau)$, *and for any* $0<|\beta|<2m$ *and any* $\mu>0$ *we can find* $C(\mu,\beta)$ *such that*

(5.10) $$\|g_{(\beta)}(x)\langle D_y\rangle^{2m}v_1\|_{-|\beta|}\leqq\mu\|Pv_1\|_0+C(\mu,\beta)\|v_1\|_0.$$

PROOF. From (5.2) and (5.4) it follows that

(5.11) $$m\tau+2\sigma m(1-\tau)-1\leqq 0.$$

For the brevity we write $v=v_1$. Let $\phi_3(t)$ be a C^∞-function in $[0,\infty)$ such that

$$\operatorname{supp}\phi_3(t)\subset[0,3),\quad \phi_3(t)\equiv 1 \quad \text{on} \quad [0,2].$$

Set $\chi_3(x,\eta)=\phi_3(g(x)h(0,\eta))$. Then we see $\chi_3(x,D_y)v=v$, since

$$\chi_3(x,D_y)v(x,y)=\int e^{iy\cdot\eta}\chi_3(x,\eta)\hat{v}(x,\eta)d\eta$$

$$=\int e^{i(x\cdot\xi+y\cdot\eta)}\chi_3(x,\eta)\chi_1(x,\xi,\eta)u\,d\xi d\eta$$

and $\chi_3(x,\eta)\equiv 1$ on $\operatorname{supp}\chi_1(x,\xi,\eta)$. Using (5.11) and the fact that $g(x)\leqq 3\langle\eta\rangle^{-2m(1-\tau)}$ on $\operatorname{supp}\chi_3(x,\eta)$, we have

$$\begin{aligned}|g\langle\eta\rangle^{2m-1+2\sigma m(1-\tau)}\chi_3|&\leqq\sqrt{3}\,\langle\eta\rangle^{m\tau-1+2\sigma m(1-\tau)}|g^{1/2}\langle\eta\rangle^m|\\&\leqq\sqrt{3}\,g^{1/2}\langle\eta\rangle^m.\end{aligned}$$

Similarly, from (5.1) and (5.11) it follows that for $0<|\beta|<2m$

$$\begin{aligned}|g_{(\beta)}\langle\eta\rangle^{2m-|\beta|}\chi_3|&\leqq C_\beta|g^{1/2-\sigma|\beta}\langle\eta\rangle^{m-|\beta|}\chi_3 g^{1/2}\langle\eta\rangle^m|\\&\leqq C'_\beta\langle\eta\rangle^{m\tau-|\beta|(1-2\sigma m(1-\tau))}|g^{1/2}\langle\eta\rangle^m|\\&\leqq C'_\beta|g^{1/2}\langle\eta\rangle^m|.\end{aligned}$$

In the second inequality we used the fact that $1/2-\sigma|\beta|\geqq 0$ for $|\beta|<2m$ since σ satisfies (5.2). Hence, using $v=\chi_3 v$, we obtain

$$\begin{aligned}\|g(x)\langle D_y\rangle^{2m-1+2\sigma m(1-\tau)}v\|_0&+\sum_{0<|\beta|<2m}\|g_{(\beta)}\langle D_y\rangle^{2m}v\|_{-|\beta|}\\&\leqq C\|g^{1/2}\langle D_y\rangle^m v\|_0.\end{aligned}$$

From Conditions 1°), 2°) and 3°), clearly we have

$$\begin{aligned}\|g^{1/2}\langle D_y\rangle^m v\|_0^2&=(g(x)\langle D_y\rangle^{2m}v,\,v)\\&\leqq C(\operatorname{Re}(Pv,\,v)+\|v\|_0^2)\\&\leqq\mu\|Pv\|_0^2+C_\mu\|v\|_0^2.\end{aligned}$$

Consequently we have (5.9) and (5.10). Q. E. D.

REMARK. By the condition (5.2), i.e.

$$\begin{cases} \sigma=1/2 & \text{when } m=1 \\ 0<\sigma<\{2(m+l(m-1))\}^{-1} & \text{when } m\geqq 2, \end{cases}$$

we can choose τ in (5.4) to satisfy (5.7) and (5.11) simultaneously.

To get (5.8) and the estimates with replaced u by $\chi_2 u$ in (4.1) and (4.2), we consider an operator $\tilde{p}(x, y, D_x, D_y)$ which is obtained by modifying $p(x, y, D_x, D_y)$ in "a neighborhood of $x=0$" as follows: Set

$$\begin{aligned}\tilde{p}(x, y, \xi, \eta)&=a(x, y, \xi)+(g(x)h(\xi, \eta)+\chi_0(x, \xi, \eta))h(\xi, \eta)^{-1}b(x, y, \eta)\\ &\equiv a(x, y, \xi)+\tilde{\chi}(x, \xi, \eta)h(\xi, \eta)^{-1}b(x, y, \eta).\end{aligned}$$

Then, we have

PROPOSITION 5.3. *$\tilde{P}=\tilde{p}(x, y, D_x, D_y)\in S^{2l/\tau}_{\lambda,\rho,\delta}$ and $\tilde{p}(x, y, \xi, \eta)$ satisfies* (H)*-condition, in the following sense:*

i) *There exists a constant $C_0>0$ such that*

$$|\tilde{p}(x, y, \xi, \eta)|\geqq C_0\lambda(\xi, \eta)^{2l} \quad \text{for large } |\xi|+|\eta|. \tag{5.12}$$

ii) *For any α and β there exists a constant $C_{\alpha\beta}$ such that*

$$|\tilde{p}^{(\alpha)}_{(\beta)}(x, y, \xi, \eta)/\tilde{p}(x, y, \xi, \eta)|\leqq C_{\alpha\beta}\lambda(\xi, \eta)^{\delta\cdot\beta-\rho\cdot\alpha} \quad \text{for large } |\xi|+|\eta|, \tag{5.13}$$

where $\rho=(1, \cdots, 1, \rho_{n_1+1}, \cdots, \rho_{n_1+n_2})$, $\rho_k=\rho_0=\min(1, l/m)$

and $\delta=(\delta_1, \cdots, \delta_{n_1}, 0, \cdots, 0)$, $\delta_k=\delta_0=2l\sigma(1-\tau)/\tau$.

PROOF. Using $|h^{(\alpha)}|\leqq C_\alpha h\lambda^{-|\alpha|}$ and $\chi_0\in S^0_{\lambda,1,\delta}$, it is easy to check $\tilde{p}(x, y, \xi, \eta)\in S^{2l/\tau}_{\lambda,\rho,\delta}$. When $|\xi|\geqq|\eta|^{m\tau/l}$ the inequality (5.12) is obvious. Indeed, from Conditions 1°), 2°) and 3°)

$$|\tilde{p}(x, y, \xi, \eta)|\geqq \operatorname{Re}\tilde{p}\geqq\operatorname{Re}a\geqq C|\xi|^{2l}\geqq\frac{C}{3}\lambda^{2l}.$$

On the other hand, if $|\xi|\leqq|\eta|^{m\tau/l}$, then we have

$$\operatorname{Re}h(\xi, \eta)^{-1}b(x, y, \eta)\geqq\langle\eta\rangle^{2m(\tau-1)}\operatorname{Re}b\geqq C\langle\eta\rangle^{2m\tau}\geqq\frac{C}{2}\lambda^{2l}.$$

From the definition of χ_0 it follows that

$$\tilde{\chi}(x, \xi, \eta)=g(x)h(\xi, \eta)+\chi_0(x, \xi, \eta)\geqq 1/2 \quad \text{in } R^n_{x,y}\times R^n_{\xi,\eta}.$$

Hence (5.12) is valid in this case, too. We check (5.13) in dividing it into two cases.

1) The case $x\in\Omega_1=\{x\,;\,g(x)h(\xi,\eta)\leqq 2\}$: We have for any α and β

$$|\tilde{p}^{(\alpha)}_{(\beta)}/\tilde{p}|\leqq|a^{(\alpha)}_{(\beta)}/\tilde{p}|+\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} C_{\alpha'\alpha''\beta'\beta''}|(\tilde{\chi}h^{-1})^{(\alpha')}_{(\beta')}b^{(\alpha'')}_{(\beta'')}/\tilde{p}|. \tag{5.14}$$

Let $\alpha=(\alpha_1,\alpha_2)$ where α_1 and α_2 are multi-indices with respect to x and y, respectively. Using (5.12) and noting that $a^{(\alpha)}_{(\beta)}=0$ if $|\alpha_2|\neq 0$ or $|\alpha_1|>2l$, the first term on the right side is bounded by $C\lambda^{-|\alpha|}$. By means of (5.1) and $g(x)\leqq 2h(\xi,\eta)^{-1}$ on Ω_1, we have

$$|(\tilde{\chi}h^{-1})^{(\alpha')}_{(\beta')}|\leqq C\lambda^{-2l(1-\tau)/\tau+\delta\cdot\beta'-|\alpha'|}.$$

If $\alpha''=(\alpha_1'',\alpha_2'')$ with $|\alpha_2''|>2m$ or $|\alpha_1''|>0$, then $b^{(\alpha'')}_{(\beta'')}=0$. When $\alpha''=(0,\alpha_2'')$ such that $|\alpha_2''|\leqq 2m$, we obtain

$$\begin{aligned}|(\tilde{\chi}h^{-1})^{(\alpha')}_{(\beta')}b^{(\alpha'')}_{(\beta'')}/\tilde{p}|&\leqq C\lambda^{-2l(1-\tau)/\tau+\delta\cdot\beta'-|\alpha'|}|\eta|^{2m-|\alpha''|}\lambda^{-2l}\\ &\leqq C\lambda^{\delta\cdot\beta-|\alpha'|-l|\alpha''|/(m\tau)}.\end{aligned}$$

From (5.4) it follows that $l/(m\tau)>1$. Hence the second term on the right side of (5.14) is bounded by $C\lambda^{\delta\cdot\beta-|\alpha|}$.

2) The case $x\in\Omega_2=\{x\,;\,g(x)h(\xi,\eta)>2\}$: Clearly we have $\tilde{p}=p$, so we have for any α and β

$$|\tilde{p}^{(\alpha)}_{(\beta)}/\tilde{p}|\leqq|a^{(\alpha)}_{(\beta)}/p|+\sum_{\beta'+\beta''=\beta} C_{\beta'\beta''}|g_{(\beta')}b^{(\alpha)}_{(\beta'')}/p|.$$

In the same way as in 1) the first term is bounded by $C\lambda^{-|\alpha|}$. Since $g(x)\in\mathcal{B}$, we may assume that $|g(x)|\leqq M$. For $\alpha=(0,\alpha_2)$ such that $|\alpha_2|\leqq 2m$, we have from (5.1)

$$\begin{aligned}|g_{(\beta')}b^{(\alpha)}_{(\beta'')}/p|&\leqq Cg^{1-\sigma|\beta'|}|\eta|^{2m-|\alpha_2|}\Big/\Big(\frac{g}{M}|\xi|^{2l}+g\langle\eta\rangle^{2m}\Big)\\ &\leqq C(M)g^{-\sigma|\beta'|}((|\xi|^{2l}+\langle\eta\rangle^{2m})^{1/2l})^{(-l+m)|\alpha_2|}\\ &\leqq C(M)g^{-\sigma|\beta'|}\lambda^{-\rho\cdot\alpha}.\end{aligned}$$

Using that $g(x)^{-1}<\dfrac{1}{2}h(\xi,\eta)$ in Ω_2, the left side is bounded by $C\lambda^{\delta\cdot\beta'-\rho\cdot\alpha}$. Thus the proposition is proved. Q. E. D.

In the consequence of Propositions 2.7 and 5.3 we have a parametrix $Q\in S^{-2l}_{\lambda,\rho,\delta}$ such that for $\tilde{P}\in S^{2l/\tau}_{\lambda,\rho,\delta}$

$$I=Q\tilde{P}+K,\quad K\in S^{-\infty}, \tag{5.15}$$

furthermore

$$(5.16)\qquad \begin{cases} Q=Q_0Q_1, \quad Q_0\in S^{-2l}_{\lambda,\rho,\delta}, \quad Q_1\in S^0_{\lambda,\rho,\delta}, \\ \sigma(Q_0)=\tilde{p}(x,y,\xi,\eta)^{-1} \qquad \text{for large} \quad |\xi|+|\eta|. \end{cases}$$

PROPOSITION 5.4. *For any β and $N>0$ there exists a constant $C=C(\beta, N)$ such that*

$$(5.17)\qquad \|g_{\langle\beta\rangle}(x)\langle D_y\rangle^{2m}u\|_{-|\beta|,\lambda}\leqq C(\|\tilde{P}u\|_{\delta\cdot\beta-|\beta|,\lambda}+\|u\|_{-N,\lambda}), \quad u\in C_0^\infty(R^n),$$

where

$$\|u\|_{s,\lambda}=\Big(\int\lambda(\xi,\eta)^{2s}|\hat{u}(\xi,\eta)|^2d\xi d\eta\Big)^{1/2}.$$

PROOF. In the same way as in the proof of Proposition 5.3, by checking the symbol of $g_{\langle\beta\rangle}(x)\langle D_y\rangle^{2m}Q_0$ we have $g_{\langle\beta\rangle}(x)\langle D_y\rangle^{2m}Q_0\in S^{\delta\cdot\beta}_{\lambda,\rho,\delta}$. From (5.16) it follows that $g_{\langle\beta\rangle}(x)\langle D_y\rangle^{2m}Q\in S^{\delta\cdot\beta}_{\lambda,\rho,\delta}$. By means of (5.15) we have

$$g_{\langle\beta\rangle}\langle D_y\rangle^{2m}=g_{\langle\beta\rangle}\langle D_y\rangle^{2m}QP+K', \quad K'\in S^{-\infty}.$$

Therefore, by Proposition 2.5 we obtain (5.17). Q. E. D.

COROLLARY. *Set $v_2=\chi_2(x, D_x, D_y)u$ for $u\in C_0^\infty(R^n)$. Then, we can find C_0 such that*

$$(5.18)\qquad \|g(x)\langle D_y\rangle^{2m}v_2\|_0\leqq C_0(\|Pv_2\|_0+\|v_2\|_0),$$

and for any $\mu>0$ and any β ($\neq 0$) we can find $C(\mu, \beta)$ such that

$$(5.19)\qquad \|g_{\langle\beta\rangle}(x)\langle D_y\rangle^{2m}v_2\|_{-|\beta|}\leqq\mu\|Pv_2\|_0+C(\mu,\beta)\|v_2\|_0, \quad (|\beta|\neq 0).$$

PROOF. Noting that $\|\cdot\|_{0,\lambda}=\|\cdot\|_0$ and that

$$\tilde{P}\chi_2=P\chi_2+K_1\chi_2 \qquad \text{for some} \quad K_1\in S^{-\infty},$$

from (5.17) with $\beta=0$ we have (5.18). Using that $\|\cdot\|_{-|\beta|}\leqq\|\cdot\|_{-|\beta|,\lambda}$ and interpolation inequality $\|\cdot\|_{\delta\cdot\beta-|\beta|,\lambda}\leqq\mu\|\cdot\|_{0,\lambda}+C_\mu\|\cdot\|_{-N,\lambda}$ ($|\beta|\neq 0$), we have (5.19) from (5.17). Q. E. D.

PROPOSITION 5.5. *For any real s and $N>0$ there exists a constant $C=C(s, N)$ such that*

$$(5.20)\qquad \|[P,\chi_j]u\|_{s,\lambda}\leqq C(\|Pu\|_{s-\tau_0,\lambda}+\|u\|_{-N,\lambda}), \quad u\in C_0^\infty(R^n),$$

where

$$\tau_0=\min_{1\leqq j\leqq n}(\rho_j-\delta_j)=\min(1-2l\sigma(1-\tau)/\tau,\ l/m).$$

PROOF. In the same way as in the proof of (5.13), by expanding the symbol of $P^{(\alpha)}_{(\beta)}Q$ we get $P^{(\alpha)}_{(\beta)}Q \in S^{\delta\cdot\beta-\rho\cdot\alpha}_{\lambda,\rho,\delta}$. It follows from (5.15) that

$$\|P^{(\alpha)}_{(\beta)}u\|_{s,\lambda} \leqq C(\|\tilde{P}u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}+\|u\|_{-N,\lambda}), \quad u\in C_0^\infty(R^n).$$

If $\tilde{\alpha}+\tilde{\beta}\neq 0$, then $\tilde{P}\chi_j{}^{(\tilde{\beta})}_{(\tilde{\alpha})} \equiv P\chi_j{}^{(\tilde{\beta})}_{(\tilde{\alpha})}$ $(j=1, 2)$ (mod $S^{-\infty}$). Consequently, for any α, β, $\tilde{\alpha}$ and $\tilde{\beta}$ $(\tilde{\alpha}+\tilde{\beta}\neq 0)$ we obtain

$$\|P^{(\alpha)}_{(\beta)}\chi_j{}^{(\tilde{\beta})}_{(\tilde{\alpha})}u\|_{s,\lambda} \leqq C(\|P\chi_j{}^{(\tilde{\beta})}_{(\tilde{\alpha})}u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}+\|u\|_{-N,\lambda}), \quad u\in C_0^\infty(R^n). \tag{5.21}$$

For the brevity we write $\chi=\chi_j$. Using the expansion formula (2.10) in Proposition 2.4, we have from (5.21)

$$\begin{aligned}
\|[P, \chi]u\|_{s,\lambda} &\leqq C\Big(\sum_{0<|\alpha+\beta|<N_0}\|P^{(\alpha)}_{(\beta)}\chi^{(\beta)}_{(\alpha)}u\|_{s,\lambda}+\|u\|_{-N,\lambda}\Big)\\
&\leqq C\Big(\sum_{0<|\alpha+\beta|<N_0}\|P\chi^{(\beta)}_{(\alpha)}u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}+\|u\|_{-N,\lambda}\Big)\\
&\leqq C\Big(\sum_{0<|\alpha+\beta|<N_0}{}'(\|\chi^{(\beta)}_{(\alpha)}Pu\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}\\
&\qquad +\|[P, \chi^{(\beta)}_{(\alpha)}]u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda})+\|u\|_{-N,\lambda}\Big)\\
&\leqq C\Big(\|Pu\|_{s-\tau_0,\lambda}+\sum_{\substack{1\leqq|\alpha+\beta|<N_0\\ 1\leqq|\tilde{\alpha}+\tilde{\beta}|<N_0}}\|P^{(\tilde{\alpha})}_{(\tilde{\beta})}\chi^{(\beta+\tilde{\beta})}_{(\alpha+\tilde{\alpha})}u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}\\
&\qquad +\|u\|_{-N,\lambda}\Big), \quad u\in C_0^\infty(R^n).
\end{aligned}$$

Here $N_0=(2l/\tau+s+N)/\tau_0$. We apply (5.21) to the second term on the last side. Repeating this procedure M times, we finally obtain

$$\begin{aligned}
\|[P, \chi]u\|_{s,\lambda} &\leqq C\Big(\|Pu\|_{s-\tau_0,\lambda}+\sum_{M\leqq|\alpha+\beta|<MN_0}\|P\chi^{(\beta)}_{(\alpha)}u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}\\
&\qquad +\|u\|_{-N,\lambda}\Big), \quad u\in C_0^\infty(R^n).
\end{aligned}$$

If we take M so that $2l/\tau+s-\tau_0M<-N$, then we get (5.20). Q. E. D.

REMARK. Clearly we obtain (5.8) from (5.20).

THE COMPLETION OF THE PROOF OF LEMMA 4.1. By Proposition 5.2 and Corollary to Proposition 5.4 we have for $\kappa=2\sigma m(1-\tau)>0$

$$\begin{aligned}
&\|g(x)\langle D_y\rangle^{2m-1+\kappa}u\|_0 \\
&\leqq \|g(x)\langle D_y\rangle^{2m-1+\kappa}v_1\|_0+\|g(x)\langle D_y\rangle^{2m}v_2\|_0 \\
&\leqq C\sum_{j=1}^{2}(\|Pv_j\|_0+\|v_j\|_0) \\
&\leqq C\Big(\|Pu\|_0+\sum_{j=1}^{2}\|[P,\chi_j]u\|_0+\|v_j\|_0\Big).
\end{aligned}$$

Using (5.8) and $\|v_j\|_0\leqq C\|u\|_0$ we obtain (4.1). Similarly, we have (4.2), too.

Q.E.D.

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On Hypoelliptic Operators with Multiple Characteristics of Odd Order

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Introduction. In the recent paper [4], Hörmander has clarified Egorov's work [3] on sub-elliptic operators, by improving several points. The purpose of the present paper is to show that the method in [4] is applicable to certain pseudo-differential operators with multiple characteristics of odd order. Rubinstein [13], Wenston [15][16], Popivanov [12] [17] and Menikoff [6] independently treated some class of differential (or pseudodifferential) operators with double (or multiple, see [17]) characteristics satisfying the conditions similar to those given by Nirenberg-Treves [10] for operators of principal type.

It should be noted that, roughly speaking, operators considered in those papers can be reduced micro-locally to $D_{x_1} + ix_1^k D_{x_2}$ (k integer), which was studied by Mizohata [8]. On the other hand, the operator considered in the present paper can not be reduced only to Mizohata type evrywhere in the sense of micro-local. At some point it will be reduced even to Egorov type $D_{x_1} + i(x_1^s D_{x_2} + x_1^a x_2^b |D_x|)$, where $|D_x|$ denotes the square root of $D_{x_1}^2 + D_{x_2}^2 + D_{x_3}^2$ and s, a,b are integers.

The plan of this paper is as follows. In section 1 we state the assumptions and result. In section 2 we reduce the proof of main theorem to "sub-elliptic estimate" for a localized operator whose symbol has a parameter $0 < \lambda \leq 1$ (see (2.37) and (3.4)). To prove this estimate, in section 3 we show that we can use the same method as in [4]. The most part of section 3 is devoted to show that the symbol of the localized operator satisfies inequalities similar to those in [4,Section 4]. In final section we prove the non-hypo-ellipticity of some operator in order to show the importance of the notion of modified-null-bicharacteristic curve, which is introduced in section 1.

1. Assumptions and Result.

We say that $p(x,\xi) \in \mathcal{B}^{\infty}(R^n_x \times R^n_\xi)$ belongs to $\overline{S}^m$ when $p(x,\xi)$ is positively homogeneous of degree m in $|\xi| \geqq 1/2$. (Clearly $\overline{S}^m$ is the subset of $S^m = S^m_{1,0}$. We refer the definition of $S^m_{1,0}$ to Kumano-go [5, p. 50].) For a conic set $U \subset R^n_x \times R^n_\xi$ and $q(x,\xi) \in C^{\infty}(U)$ with positive homogeneous of degree m in $|\xi| \geqq 1/2$ we write $q(x,\xi) \in \overline{S}^m(U)$.

Let $p_1(x,\xi)$ belong to $\overline{S}^1$ and be real principal type, that is, be real valued and satisfy

$$(1.1) \qquad d_{x\xi} p_1(x,\xi) \neq 0 \quad \text{on} \quad \Gamma = p_1^{-1}(0) \cap \{|\xi| \geqq 1/2\} .$$

Let ℓ be an odd integer $\geqq 3$ and let $a(x,\xi) \in \overline{S}^{\ell-1}$ be complex valued and satisfy

$$(1.2) \qquad \operatorname{Re} a \neq 0 \quad \text{on } \Gamma$$

and moreover

$$(1.3) \qquad H_{p_1} \operatorname{Re} a = 0 \quad \text{on } \Gamma$$

where H_{p_1} denotes the Hamilton vector field of p_1.

Then under certain conditions among p_1, Re a and Im a we shall discuss the hypoellipticity for a pseudodifferential operator L of order ℓ which has the form

$$(1.4) \qquad L = P(x,D_x) + A(x,D_x) \quad \text{in } R^n_x ,$$

$$\sigma(P) = (p_1(x,\xi))^{\ell} , \qquad \sigma(A) = a(x,\xi) .$$

Here $\sigma(P)$ denotes the symbol of pseudodifferential operator $P(x,D_x)$.

First we assume that

(1.5) for any $(x_0,\xi_0) \in \Gamma$ there exist a conic neighborhood U of (x_0,ξ_0) and $q_0(x,\xi) \in \overline{S}^0(U)$ such that

(1.6) $$H_{p_1} q_0 = 1 \quad \text{in} \quad U$$

and for $j = 1,\dots, \ell-2$

(1.7) $$H_{q_0}^j a = 0 \quad \text{on} \quad \Gamma \cap U.$$

To state the second condition corresponding to (A) in Egorov[3], or (Ψ) in Nirenberg - Treves[10], we define the modified-null - bicharacteristic curve of p_1 through $(x_0,\xi_0) \in \Gamma$ by the curve

(1.8) $$[-T,T] \ni t \longmapsto (x(t), \xi(t)),$$

where $(x(t), \xi(t))$ is the solution to

(1.9) $$dx/dt = d_\xi(p_1 + \sqrt[\ell]{\mathrm{Re}\ a})$$
$$d\xi/dt = -d_x(p_1 + \sqrt[\ell]{\mathrm{Re}\ a}), \quad (x(0),\xi(0)) = (x_0,\xi_0),$$

and $\sqrt[\ell]{\mathrm{Re}\ a}$ denotes a unique real ℓ power root of Re a. It follows from (1.3) that if $(x_0,\xi_0) \in \Gamma$, then $(x(t),\xi(t)) \in \Gamma$. The right hand side of (1.9) are not homogeneous in ξ, so that the behavior of modified - null - bicharacteristic curve is not so. But we can define it on $[-T,T]$ for some $T > 0$ uniformly if $(x_0,\xi_0) \in \Gamma$ varies in a compact conic set ($|\xi_0| > 1$), because $d_\xi \sqrt[\ell]{\mathrm{Re}\ a}$ and $|\xi|^{-1} d_x \sqrt[\ell]{\mathrm{Re}\ a}$ are $O(|\xi|^{-1/\ell})$.

Second condition is that

(1.10) for any $(x_0,\xi_0) \in \Gamma$ with $|\xi_0|$ sufficiently large, if $\operatorname{Im} a(x(t_0),\xi(t_0)) > 0$ for some $t_0 \in [-T,T]$, then $\operatorname{Im} a(x(t),\xi(t)) \geq 0$ for all $t \in (t_0,T]$, where $(x(t),\xi(t))$ is the modified-null-bicharacteristic curve of p_1 through (x_0,ξ_0).

For a multi-index $I = (i_1,\ldots,i_k)$ whose components i_j are 1 or 2, we use the following notations: $|I| = k$, $b(I) =$ " the number of j such that $i_j = 2$ ", $J = \{ I ; b(I) \leq \ell-1 \}$, $\mu(I) = (\ell-1)|I|/(\ell-b(I))$. For a $\mu > 0$ we put $J_\mu = \{ I \in J ; \mu(I) \leq \mu \}$. Set $Q_0 = \{ \mu(I) ; I \in J \}$. Then we can write $Q_0 = \{ \mu_j ; j = 1,2,\ldots \}$ with an appropriate increasing sequence of rational numbers. For any $(x,\xi) \in \Gamma$, $\mu(x,\xi)$ denotes the rational $\mu_j \in Q_0$ such that $p_I(x,\xi) \not\equiv 0$ for some $I \in J_{\mu_j}$ and $p_I(x,\xi) = 0$ for any $I \in J_{\mu_{j-1}}$, where

$$p_I(x,\xi) = H_{i_1} H_{i_2} \cdots H_{i_{k-1}} p_{i_k}(x,\xi) , \tag{1.11}$$

$H_1 = H_{p_1}$, $H_2 = H_{p_2}$ and $p_2(x,\xi) = \operatorname{Im} a(x,\xi)$. The third condition corresponding to (B) in Egorov [3] is that

(1.12) for all $(x,\xi) \in \Gamma$ there exists some $\mu \in Q_0$ such that $\mu(x,\xi) \leq \mu < \infty$.

In what follows we denote the norm in Sobolev space H^s by $\|\cdot\|_s$. We write $\|\cdot\| = \|\cdot\|_0$.

Theorem 1.1. Let (1.1)-(1.3) hold and let $L(x,D_x)$ in (1.4) satisfy (1.5)-(1.7), (1.10) and (1.12). Then for any compact set K of R^n there exists a constant C_K such that

$$(1.13)\quad \|u\|_{\ell-2+\sigma'} + \|P'u\|_\sigma \leq C_K(\|Lu\| + \|u\|), \quad u \in C_0^\infty(K),$$

where $\sigma(P') = (p_1(x,\xi))^{\ell-1}$, $\sigma = (\ell-1)/\ell\mu$, $\sigma' = \sigma + 1/\ell$.

Remark 1. It follows from (1.13) that P is hypoelliptic. See Oleinik-Radkevich [11] and Morimoto [9, Theorem 2.2]. Theorem 2.2 of [9] is stated only for differential operators but its proof is also applicable to pseudo-differential operators.

Remark 2. In differential operators we have the following example :

$$D_{x_1}^5 + D_{x_2}^4 + D_{x_3}^4 + i(x_1^2 x_2 (D_{x_2}^4 + D_{x_3}^4) + x_1^3 D_{x_2}^2 (D_{x_2}^2 + D_{x_3}^2)) \text{ in } R_x^3.$$

All conditions of Theorem 1.1 are satisfied. Specially, the condition (1.10) is satisfied at $(0,\xi_0)$ with $\xi_0 = (0,0,\xi_{03})$ as follows; the sign of Im a changes from - to + along the modified - null - bicharacteristic curve of p_1 through $(0,\xi_0)$. At the point $(0,\xi_0)$ this example must be reduced to Egorov type. Note that Egorov's operator of principal type is not the differential operator but the pseudodifferential operator. (See the introduction of [3].) For this example we have $\sigma = 1/25$.

Remark 3. The condition (1.10) is delicated and necessary in general for the hypoellipticity of P. Indeed, the operator with replaced $\operatorname{Re} a = \xi_2^4 + \xi_3^4$ in the above example by $(\xi_2 - \xi_3)^4 + \xi_3^4$ satisfies all conditions except (1.10), which is violated at $(0,\xi_0)$ with $\xi_0 = (0,0,\xi_{03})$ and $\xi_{03} > 0$. Furtheremore, we have

Proposition 1.2. Differential operator $\tilde{L}$

$$\tilde{L} = D_{x_1}^5 + (D_{x_2} - D_{x_3})^4 + D_{x_3}^4 + i(x_1^2 x_2 (D_{x_2}^4 + D_{x_3}^4) + x_1^3 D_{x_2}^2 (D_{x_2}^2 + D_{x_3}^2)) \quad \text{in } R_x^3, \tag{1.14}$$

is not hypoelliptic at the origin.

2. Reduction to Localized Operator.

Let $h(x) \in C_0^\infty(R_x^n)$ be 1 for $|x| < 1/2$ and vanish for $|x| > 1$. Set $h_\varepsilon(x) = h(x/\varepsilon)$ for a small $\varepsilon > 0$. For a $f(x,\xi) \in \overline{S}^m$ and $\gamma = (x_0, \overline{\xi}_0) \in R^n \times S^{n-1}$ we introduce a pseudodifferential operator $F_{\gamma,\varepsilon,\lambda}$ with a parameter $0 < \lambda \leq 1$ and a small $\varepsilon > 0$ as follows:

$$(2.1)\qquad F_{\gamma,\varepsilon,\lambda}(y,D_y)v = \lambda^{-2m}\int e^{iy\eta} h_\varepsilon(\lambda y) f(x_0+\lambda y, \overline{\xi}_0+\lambda\eta) h_\varepsilon(\lambda\eta)\hat{v}(\eta)d\!\!\!^{-}\eta \ ,$$

$$v \in S_y \ , \qquad d\!\!\!^{-}\eta = (2\pi)^{-n}d\eta \ ,$$

where $\hat{v}$ denotes the Fourier transform of v.
Obviously, for a fixed $\varepsilon > 0$, $\{ \lambda^{2m}\sigma(F_{\gamma,\lambda,\varepsilon})(y,\eta);\ 0 < \lambda \leq 1 \}$ is a bounded set of $S^0_{0,0}$.
Furthermore we obtain for a sufficiently small $\varepsilon > 0$

$$(2.2)\qquad (F_{\gamma,\varepsilon,\lambda}v)(\lambda^{-1}(x-x_0)) = e^{-i\lambda^{-2}x\cdot\overline{\xi}_0} h_\varepsilon(x-x_0) f(x,D_x) h_\varepsilon(\lambda^2 D_x - \overline{\xi}_0) u(x) \ ,$$

where $\hat{u}(\xi) = \lambda^n \hat{v}(\lambda(\xi-\lambda^{-2}\overline{\xi}_0))\exp(ix_0(\lambda^{-2}\overline{\xi}_0 - \xi))$.

Lemma 2.1. For any compact set $K \subset R^n$ there exists a constant C_K such that (1.13) holds if and only if for any $\gamma = (x_0, \overline{\xi}_0) \in R^n \times S^{n-1}$ one can find positives $\varepsilon_i = \varepsilon_{i,\gamma}$ ($i=1,2,3$, $\varepsilon_1 < \varepsilon_2 < \varepsilon_3$) and a constant C_γ so that for any $0 < \lambda \leq 1$ the following estimate holds ;

$$(2.3)\qquad \lambda^{-2(\ell-2+\sigma')} \| H_{\varepsilon_1,\lambda} v \| + \lambda^{-2\sigma} \| P'_{\gamma,\varepsilon_1,\lambda} v \|$$

$$\leq C_\gamma (\| (P_{\gamma,\varepsilon_2,\lambda} + A_{\gamma,\varepsilon_2,\lambda}) v \| + \| P'_{\gamma,\varepsilon_3,\lambda} v \|$$

$$+ \lambda^{-2(\ell-2+\sigma)} \| v \|), \quad v \in S_y ,$$

where $H_{\varepsilon,\lambda}, P_{\gamma,\varepsilon,\lambda}, P'_{\gamma,\varepsilon,\lambda}$ and $A_{\gamma,\varepsilon,\lambda}$ are defined by (2.1) with f replaced by 1, p_1^{ℓ}, $p_1^{\ell-1}$ and a, respectively.

Proof. In view of (2.2) it is not difficult to see the necessity of (2.3). We only show the sufficiency. The proof is the same way as in [4, p. 143], except the appearance of the second term in left hand side of (1.13) or (2.3). Note that for any $\overline{\xi}_0 (|\overline{\xi}_0| = 1)$ and any small $\varepsilon > 0$ and any real s

$$(2.4)\qquad C^{-1} \| h_\varepsilon(\lambda D_y) v \| \leq \| (\overline{\xi}_0 + \lambda D_y)^s h_\varepsilon(\lambda D_y) v \|$$

$$\leq C \| h_\varepsilon(\lambda D_y) v \|, \quad v \in S_y ,$$

holds for some $C = C_{s,\varepsilon}$ since $C^{-1} \leq |\overline{\xi}_0+\xi|^s \leq C$ on supp $h_\varepsilon(\xi)$. Substituting $\hat{v}(\eta) = h_{\varepsilon_0}(\lambda\eta)\hat{u}(\lambda^{-1}\eta + \lambda^{-2}\overline{\xi}_0)$ $\exp(ix_0(\lambda^{-1}\eta + \lambda^{-2}\overline{\xi}_0))$ for $u \in C_0^\infty(K)$ and some $\varepsilon_0 > \varepsilon_3$ into (2.3), we obtain by means of (2.2) and (2.4)

$$\| h_1(x-x_0) h_1(\lambda^2 D_x - \overline{\xi}_0) | D_x|^{\ell-2+\sigma'} u \|$$

$$+ \| h_1(x-x_0) P'(x,D_x) |D_x|^\sigma h_1(\lambda^2 D_x - \overline{\xi}_0) u \|$$

$$\leq C(\| h_2(x-x_0)(P(x,D_x) + A(x,D_x)) h_2(\lambda^2 D_x - \overline{\xi}_0) u \|$$

$$+ \| h_3(x-x_0) P'(x,D_x) h_3(\lambda^2 D_x - \overline{\xi}_0) u \|$$

$$+ \| h_0(\lambda^2 D_x - \overline{\xi}_0) |D_x|^{\ell-2+\sigma} u \|),$$

where $h_j = h_{\varepsilon_j}$. Here we used the fact that, for a fixed $\varepsilon > 0$, $\{ \lambda^{-2}[h_\varepsilon(\lambda y), h_\varepsilon(\lambda D_y)];\ 0 < \lambda \leq 1 \}$ is a bounded set of of $S^0_{0,0}$. Since $[P, h_2(\lambda^2 D_x - \bar{\xi}_0)]$ can be estimated by the second term of the left hand side, the proof is completed from the following proposition and usual finite covering argument over $K \times S^{n-1}$.

Proposition 2.2. Let $h(\xi) \in C_0^\infty(R^n)$ be 1 in a neighborhood of 0 and let $\bar{\xi}_0$ belong to S^{n-1}. Then one can find some $\psi_j(\xi) \in \bar{S}^0$ ($j = 1,2$) such that

(2.5) $\psi_j(\bar{\xi}_0) \neq 0$, supp $\psi_j \subset$ some conic neighborhood of $\bar{\xi}_0$

and for any N we have for some constant $C > 0$

$$C^{-1} \| \psi_1(D_x) u \|^2$$

$$\leq \int_0^1 \| h(\lambda^2 D_x - \bar{\xi}_0) u \|^2 / \lambda \, d\lambda + \| u \|_{-N}$$

$$\leq C(\| \psi_2(D_x) u \|^2 + \| u \|_{-N}) , \qquad u \in S_x$$

Proof. Put $r = |\xi|$, $\theta = \xi/|\xi|$. Then

$$\int_0^1 \| h(\lambda^2 D_x - \bar{\xi}_0) u \|^2 / \lambda \, d\lambda$$

$$= \int d\theta \int_0^1 d\lambda \int_0^\infty h(\lambda^2 r\theta - \bar{\xi}_0)^2 |\hat{u}(r\theta)|^2 / \lambda \, dr.$$

It is easy to see that supp $h(\lambda^2 r\theta - \bar{\xi}_0)$ is evaluated from above and below by

$$\{ (\theta, r, \lambda) \ ; \ \theta \in \text{supp } \psi_j \cap S^{n-1} \quad \text{and}$$

$$C_j^{-1} \leq r\lambda^2 \leq C_j \ \}$$

for some $\psi_j \in \bar{S}^0$ satisfying (2.5) and some $C_j > 0$ ($j = 1,2$).

Therefore the integral is bounded by constant times

$$\int \psi_j^2(\theta)d\theta \int_{1/C}^{\infty} |\hat{u}(r\theta)|^2 \, dr \int_{(Cr)^{-1/2}}^{(C/r)^{1/2}} d\lambda \quad ,$$

where we used $(r/C)^{1/2} < 1/\lambda < (Cr)^{1/2}$ and $C = C_j$. This gives the desired estimate.

Remark. The content of this proposition is briefly stated in [4, p.143].

Since (2.3) is valid for $\gamma \notin \Gamma$, in view of Lemma 2.1 we now fix a $\gamma \in \Gamma$. Let a function $f_\lambda(y,\eta) \in \mathcal{B}^\infty(R_y^n \times R_\eta^n)$ with a parameter $0 < \lambda \leq 1$ satisfy

$$|\partial_y^\alpha \partial_\eta^\beta f_\lambda(y,\eta)| \leq C_{\alpha\beta}\lambda^{|\alpha+\beta|} \tag{2.6}$$

for any α,β , where $C_{\alpha\beta}$ is a constant independent of λ. Define a pseudodifferential operator $F_\lambda(y,D_y)$ by

$$F_\lambda v = \int e^{iy\eta} f_\lambda(y,\eta)\hat{v}(\eta)d\eta \quad , \quad v \in S \ . \tag{2.7}$$

If $f_\lambda(y,\eta)$ equals $f(\lambda y,\lambda\eta)$ for some $f(x,\xi) \in \mathcal{B}^\infty(R_x^n \times R_\xi^n)$, then we say that the operator $F_\lambda(y,D_y)$ has an original symbol $f(x,\xi)$. Under this notation, (2.3) for a fixed γ becomes

$$\begin{aligned} &\lambda^{-2(\ell-2+\sigma')} \| H_{\varepsilon_1,\lambda} v \| + \lambda^{-2(\ell-1+\sigma)} \| P'_{\varepsilon_1,\lambda} v \| \\ &\leq C(\| (\lambda^{-2\ell} P_{\varepsilon_2,\lambda} + \lambda^{-2(\ell-1)} A_{\varepsilon_2,\lambda}) v \| \\ &\qquad + \lambda^{-2(\ell-1)} \| P'_{\varepsilon_3,\lambda} v \| + \lambda^{-2(\ell-2+\sigma)} \| v \|), \end{aligned} \tag{2.8}$$

$$v \in S \ ,$$

where the original symbols of $H_{\varepsilon,\lambda}$, $P_{\varepsilon,\lambda}$, $P'_{\varepsilon,\lambda}$ and $A_{\varepsilon,\lambda}$ are $h_\varepsilon(x,\xi) = h_\varepsilon(x)h_\varepsilon(\xi)$, $h_\varepsilon(p_{1,\gamma})^\ell$, $h_\varepsilon(p_{1,\gamma})^{\ell-1}$ and $h_\varepsilon a_\gamma$, respectively. Here $p_{1,\gamma}(x,\xi) = p_1(x+x_0, \xi+\bar{\xi}_0)$, $a_\gamma(x,\xi) = a(x+x_0, \xi+\bar{\xi}_0)$.

Lemma 2.3. If (2.8) is valid and χ is a C^∞ canonical transformation keeping 0 fixed which is defined near 0, then (2.8) remains valid with some other ε_j and C if $P_{\varepsilon,\lambda}$, $A_{\varepsilon,\lambda}$ and $P'_{\varepsilon,\lambda}$ are replaced by $\tilde{P}_{\varepsilon,\lambda}$, $\tilde{A}_{\varepsilon,\lambda}$ and $\tilde{P}'_{\varepsilon,\lambda}$, respectively, whose original symbols are $h_\varepsilon(p_{1,\gamma}\circ\chi)^\ell$, $h_\varepsilon(a_\gamma\circ\chi)$ and $h_\varepsilon(p_{1,\gamma}\circ\chi)^{\ell-1}$, respectively.

As pointed out in [4, the proof of Lemma 3.2] it suffices to prove the lemma when χ has a generating function $S(x,\xi)$, that is, χ ; $(x,d_xS(x,\xi)) \mapsto (d_\xi S(x,\xi),\xi)$. The proof is based on several propositions on Fourier integral operators with phase function $S_\lambda(y,\eta) = \lambda^{-2}S(\lambda y,\lambda\eta)$.

Definition 2.4. For any $f_\lambda(y,\eta) \in B^\infty$ with (2.6) and for any $S(x,\xi) \in C^\infty(R^n_x \times R^n_\xi)$ satisfying

$$\det \partial_x\partial_\xi S(0,0) \neq 0\ ,\quad (d_xS(0,0),\ d_\xi S(0,0)) = (0,0) \tag{2.9}$$

we define the Fourier integral operator $F_{S_\lambda}(y,D_y)$ with a parameter $0 < \lambda \leq 1$ by

$$F_{S_\lambda}v = \int e^{iS_\lambda(y,\eta)} k_\lambda(y,\eta)\, f_\lambda(y,\eta)\hat{v}(\eta)d\eta\ ,\quad v \in S \tag{2.10}$$

where $S_\lambda(y,\eta) = \lambda^{-2}S(\lambda y,\lambda\eta)$ and $k_\lambda(y,\eta) = k(\lambda y,\lambda\eta)$.

Here we assume that $k(x,\xi) \in C_0^\infty$ is 1 in a neighborhood of 0 and $\det \partial_x\partial_\xi S(x,\xi) \neq 0$ on supp k. We define the conjugate Fourier integral operator $F_{S_\lambda}^*$ with a parameter $0 < \lambda \leq 1$ by

$$(2.11)\quad F_{S_\lambda}^*(y,D_y)v = \iint e^{i(y\eta - S_\lambda(\tilde{y},\eta))} k_\lambda(\tilde{y},\eta) f_\lambda(\tilde{y},\eta) v(\tilde{y})\, d\tilde{y}\, d\!\!\!^{-}\eta\,, \quad v \in S\,,$$

We call $f_\lambda(y,\eta)$ the symbol of $F_{S_\lambda}(y,D_y)$ ($F_{S_\lambda}^*(y,D_y)$), and morover if $f_\lambda(y,\eta) = f(\lambda y,\lambda\eta)$ for some $f \in B^\infty$, then we call $f(x,\xi)$ the original symbol of F_{S_λ} ($F_{S_\lambda}^*$). We write $F_{S_\lambda} = I_{S_\lambda}$, $F_{S_\lambda}^* = I_{S_\lambda}^*$ if $f = 1$.

Put

$$(2.12)\quad \tilde{d}_\xi S(x,\tilde{\xi},\xi) = \int_0^1 d_\xi S(x,\ \tilde{\xi}+\theta(\xi-\tilde{\xi}))\,d\theta$$

$$\tilde{d}_x S(\tilde{x},x,\xi) = \int_0^1 d_x S(\tilde{x}+\theta(x-\tilde{x}),\ \xi)\,d\theta$$

Put $\tilde{x} = \tilde{d}_\xi S(x,\tilde{\xi},\xi)$ and $\tilde{\xi} = \tilde{d}_x S(\tilde{x},x,\xi)$. Then the inverses

$$(2.13)\quad x = \phi(\tilde{x};\tilde{\xi},\xi) \quad \text{and} \quad \xi = \psi(\tilde{\xi};\tilde{x},x)$$

exist, respectively, in a neighborhood of 0 on account of (2.9) if supp k is sufficiently small.

Proposition 2.5. If supp k in (2.10) and (2.11) is suffuciently small, then $I_{S_\lambda} I_{S_\lambda}^*$ and $I_{S_\lambda}^* I_{S_\lambda}$ are pseudodifferential operators whose symbols are

$$(2.14)\quad \iint e^{-i\tilde{y}\tilde{\eta}} r_\lambda(y,\ \eta+\tilde{\eta},\ y+\tilde{y})\, d\tilde{y}d\!\!\!^{-}\tilde{\eta}$$

and

$$(2.15)\qquad \iint e^{-i\tilde{y}\tilde{\eta}} r_\lambda^*(\eta+\tilde{\eta},\ y+\tilde{y},\ \eta)\,d\tilde{y}d\tilde{\eta}$$

respectively, where $r_\lambda(y,\tilde{\eta},\tilde{y})$ and $r_\lambda^*(\eta,\tilde{y},\eta)$ are given by

$$(2.16)\qquad (k(x,\xi)k(\tilde{x},\xi)\,|\det \int_0^1 \partial_x\partial_\xi S(\tilde{x}+\theta(x-\tilde{x}),\xi)\,d\theta\,|^{-1})_{\xi=\psi(\tilde{\xi};\tilde{x},x)}$$

and

$$(2.17)\qquad (k(x,\xi)k(x,\tilde{\xi})\,|\det \int_0^1 \partial_x\partial_\xi S(x,\tilde{\xi}+\theta(\xi-\tilde{\xi})\,d\theta\,|^{-1})_{x=\psi(\tilde{x};\tilde{\xi},\xi)},$$

respectively. Here $(x,\tilde{x},\xi,\tilde{\xi}) = \lambda(y,\tilde{y},\eta,\tilde{\eta})$.

Proof is directly calculated by means of the change of variable ; $\tilde{\xi} = \tilde{d}_x S(\tilde{x},x,\xi)$ and $\tilde{x} = \tilde{d}_\xi S(x,\tilde{\xi},\xi)$ respectively.

Corollary 2.6. The operators F_{S_λ} and $F_{S_\lambda}^*$ for any f_λ with (2.6) are L_2-bounded uniformly with respect to $0 < \lambda \leq 1$. If ε is small enough, then $I_{S_\lambda} I_{S_\lambda}^*$ and $I_{S_\lambda}^* I_{S_\lambda}$ are elliptic on supp $h_\varepsilon(\lambda y,\lambda\eta)$, that is, the estimates

$$(2.18)\qquad \| H_{\varepsilon,\lambda} v \| \leq C_1 (\| I_{S_\lambda} I_{S_\lambda}^* v \| + \lambda^2 \| v \|),$$

$$(2.19)\qquad \| H_{\varepsilon,\lambda} v \| \leq C_2 (\| I_{S_\lambda}^* I_{S_\lambda} v \| + \lambda^2 \| v \|), \qquad v \in S\ ,$$

hold for some constants C_1 and C_2 .

Proof. The symbols of $F_{S_\lambda}^* F_{S_\lambda}$ and $F_{S_\lambda} F_{S_\lambda}^*$ are given by the versions of (2.14) and (2.15), respectively, which belong to a bounded set of $S^0_{0,0}$ uniformly on account of (2.6). The boundedness of $F_{S_\lambda}^* F_{S_\lambda}$ and $F_{S_\lambda} F_{S_\lambda}^*$ show the first statement. Note that for any $p(x,\tilde{\xi},\tilde{x},\xi) \in B^\infty(R^{4n})$

$$(2.20)\quad O_s\text{-}\iint e^{-i\tilde{x}\tilde{\xi}} p(x,\tilde{\xi},\tilde{x},\xi)\,d\tilde{x}d\tilde{\xi}$$
$$= \sum_{|\alpha|<N} p_\alpha(x,0,0,\xi)/\alpha! + N \sum_{|\beta|=N} \int_0^1 (1-\theta)^{N-1} \cdot$$
$$O_s\text{-}\iint e^{-i\tilde{x}\tilde{\xi}} p_\beta(x,\theta\tilde{\xi},\tilde{x},\xi)\,d\tilde{x}d\tilde{\xi}d\theta/\beta!$$

holds for any positive integer N, where $p_\alpha(x,\tilde{\xi},\tilde{x},\xi) = \partial_{\tilde{\xi}}^{\alpha} D_{\tilde{x}}^{\alpha} p(x,\tilde{\xi},\tilde{x},\xi)$. Here $O_s\text{-}\iint$ denotes the oscillatory integral (see [5, P.42]). Applications of (2.20) to (2.14) and (2.15) yield (2.18) and (2.19), respectively.

Proposition 2.7. Let $F_\lambda(y,D_y)$ be the pseudodifferential operator with original symbol $f(x,\xi) \in B^\infty$. Let G_{S_λ} and $\tilde{G}_{S_\lambda}$ be Fourier integral operators whose original symbols are $f(x, d_x S(x,\xi))$ and $f(d_\xi S(x,\xi),\xi)$, respectively. Then $\lambda^{-2}(F_\lambda I_{S_\lambda} - G_{S_\lambda})$ and $\lambda^{-2}(I_{S_\lambda} F_\lambda - \tilde{G}_{S_\lambda})$ are L_2-bounded operators uniformly with respect to $0 < \lambda \leq 1$.

Proof. It is easy to check that the equations

$$F_\lambda I_{S_\lambda} v = \int e^{iS_\lambda(y,\eta)} r_\lambda(y,\eta)\hat{v}(\eta)\,d\!\!\!^{-}\eta$$

and

$$I_{S_\lambda} F_\lambda v = \int e^{iS_\lambda(y,\eta)} \tilde{r}_\lambda(y,\eta)\hat{v}(\eta)\,d\!\!\!^{-}\eta\ ,$$

for

$$(2.21)\quad r_\lambda(y,\eta) = O_s\text{-}\iint e^{-i\tilde{y}\tilde{\eta}} f(x,\tilde{\xi}+\tilde{d}_x S(x,x+\tilde{x},\xi)k(x+\tilde{x},\xi)\,d\tilde{y}d\!\!\!^{-}\tilde{\eta}$$

and

$$(2.22)\quad \tilde{r}_\lambda(y,\eta) = O_s\text{-}\iint e^{-i\tilde{y}\tilde{\eta}} f(\tilde{x}+\tilde{d}_\xi S(x,\xi+\tilde{\xi},\xi),\xi)k(x,\xi+\tilde{\xi})\,d\tilde{y}d\!\!\!^{-}\tilde{\eta}\ ,$$

respectively, where $(x,\tilde{x},\xi,\tilde{\xi}) = \lambda(y,\tilde{y},\eta,\tilde{\eta})$. Applications of (2.20) to (2.21) and (2.22) complete the proof.

Corollary 2.8. For $f(x,\xi) \in C_0^\infty$ with support contained a sufficiently small neighborhood of 0 , set $\tilde{f}(x,\xi) = (f\circ\chi)(x,\xi)$, where χ is defined by

$$(x,d_x S(x,\xi)) \to (d_\xi S(x,\xi),\xi) .$$

Then $\lambda^{-2}(F_\lambda I_{S_\lambda} - I_{S_\lambda}\tilde{F}_\lambda)$ is L_2-bounded uniformly for $0 < \lambda \leqq 1$. Furthermore, if $f(x,\xi) = (p_1(x,\xi))^\ell h_\varepsilon(x,\xi)$ then for $\tilde{f} = (p_1{}^\ell h_\varepsilon)\circ\chi$, the estimate

$$(2.23)\quad \|(F_\lambda I_{S_\lambda} - I_{S_\lambda}\tilde{F}_\lambda)v\| \leqq C(\lambda^2\|F_\lambda' I_{S_\lambda} v\| + \lambda^2\|I_{S_\lambda}\tilde{F}_\lambda' v\|), \qquad v \in S ,$$

holds for some constant C , where the original symbols of F_λ' and $\tilde{F}_\lambda'$ are $(p_1)^{\ell-1}h_{\varepsilon'}$ and $((p_1)^{\ell-1}h_{\varepsilon'})\circ\chi$ for some $\varepsilon' > \varepsilon$.

Proof. The first part follows from Proposition 2.7. The second part is obtained by checking the second terms of the expansions of (2.21) and (2.22).

Remark. It is clear that $\lambda^{-2}(I_{S_\lambda}{}^* F_\lambda - F_\lambda I_{S_\lambda}{}^*)$ is L_2 - bounded operator uniformly for $0 < \lambda \leqq 1$. Indeed, this follows from $\|F_\lambda I_{S_\lambda} - I_{S_\lambda}\tilde{F}_\lambda\| = \|I_{S_\lambda}{}^* F_\lambda^* - \tilde{F}_\lambda^* I_{S_\lambda}{}^*\|$ and the fact that $\lambda^{-2}(F_\lambda - F_\lambda^*)$ and $\lambda^{-2}(\tilde{F}_\lambda - \tilde{F}_\lambda^*)$ are L_2 - bounded.

Proof of Lemma 2.3. Taking $I_{S_\lambda} v$ as v in (2.8) and noting Corollary 2.8, we obtain (2.8) for operators transformed, by using the fact that for any small $\varepsilon > 0$ the eatimate

$$C^{-1} \|H_{\varepsilon',\lambda} v\| \leq \|I_{S_\lambda} H_{\varepsilon,\lambda} v\| + \lambda^2 \|v\|$$
$$\leq C(\|H_{\varepsilon'',\lambda} v\| + \lambda^2 \|v\|) \quad , \qquad v \in S \ ,$$

holds for some $0 < \varepsilon' < \varepsilon < \varepsilon''$ and some constant C, which follows from (2.19).

Now we take a canonical transformation χ such that $p_{1,\gamma} \circ \chi = \xi_1$ and $q_{0,\gamma} \circ \chi = x_1$, where $q_{0,\gamma}$ is defined from q_0 given in (1.5) by the same way as in $p_{1,\gamma}$. Darboux theorem (see [7,Proposition 2.1]) shows that (1.6) guarantees the existence of such a χ. Application of Lemma 2.3 gives

Lemma 2.9. The estimate (2.8) is valid if for some ε and ε' $(\varepsilon' > \varepsilon > 0)$ the estimate

$$\text{(2.24)} \quad \lambda^{-2(\ell-2+\sigma')} \|h_\varepsilon(\lambda y', \lambda D_{y'}) v\| + \lambda^{-2\sigma} \|D_{y_1}^{\ell-1} h_\varepsilon(\lambda y', \lambda D_{y'}) v\|$$
$$\leq C(\|(D_{y_1}^{\ell} + \lambda^{-2(\ell-1)} \tilde{A}_{\varepsilon',\lambda}(y, D_{y'})) v\|$$
$$+ \|D_{y_1}^{\ell-1} v\| + \lambda^{-2(\ell-2+\sigma)} \|v\|) \ ,$$

$$\text{if } v \in S \text{ vanishes for } |y_1| > \varepsilon \ ,$$

holds for some C, where the symbol of $\tilde{A}_{\varepsilon,\lambda}(y, D_{y'})$ is $\tilde{a}(y_1, \lambda y', \lambda\eta') h_\varepsilon(\lambda y', \lambda\eta')$. Here $\tilde{a}(x, \xi') = (a_\gamma \circ \chi)(x, 0, \xi')$.

Proof. Condition (1.7) leads us to the

$$(2.25)\qquad a_\gamma\circ\chi\,(x,\xi) = a_\gamma\circ\chi(x,0,\xi') + b(x,\xi)\xi_1^{\ell-1}$$

for some b near origin since H_{q_0} is transformed to ∂_{ξ_1}. Therefore, substituting $h_\varepsilon(y_1, \lambda^2 D_{y_1})h_{\varepsilon''}(\lambda y', \lambda D_{y'})v$ ($\varepsilon < \varepsilon'' < \varepsilon'$) for v of (2.24) with replaced $\tilde{a}$ by $a_\gamma\circ\chi(x,\xi)$ and changing variable (y_1, η_1) into $(\lambda y_1, \lambda^{-1}\eta_1)$ we get the estimate (2.8) canonically transformed.

Proposition 2.10. Let $g_\lambda(y,\eta) \in C^\infty$ with a parameter $0 < \lambda \leqq 1$ satisfy for any α, β

$$(2.26)\qquad |\partial_\eta^\alpha \partial_y^\beta g_\lambda| \leqq C_{\alpha\beta}(|\eta_1|^2 + \lambda^{-4\delta})^{(m-|\alpha_1|)/2}\lambda^{|\alpha'+\beta'|}$$

where $0 < \delta \leqq 1$ and m integer. Assume that, for some $h_\lambda(y',\eta') = h(\lambda y', \lambda\eta')$, g_λ satifies for some $c_0 > 0$

$$(2.27)\qquad |g_\lambda| \geqq c_0(|\eta_1|^2 + \lambda^{-4\delta})^{m/2} \quad \text{on} \quad \{|y_1| < 1\} \times \operatorname{supp} h_\lambda .$$

Then we get

$$(2.28)\qquad \|D_{y_1}^m h_\lambda(y',D_{y'})v\| + \lambda^{-2\delta m}\|h_\lambda(y',D_{y'})v\| \leqq C(\|G_\lambda(y,D_y)v\| + \|v\|) ,$$

if $v \in S$ vanishes $|y_1| > 1$.

Proof is omitted. (for example, see [5,p. 77]).

Applying this proposition with $m = \ell$ and $\delta = (\ell-1)/\ell$ to $\eta_1^\ell + \lambda^{-2(\ell-1)}\tilde{a}(y_1,\lambda y',\lambda\eta')h_\varepsilon(\lambda y',\lambda\eta')$, we obtain (2.24) if $\operatorname{Im}\tilde{a}(0) \neq 0$. From now on we assume $\operatorname{Im}\tilde{a}(0) = 0$. Let $\omega(x,\xi')$ be a ℓ power root of $(\tilde{a}h_\varepsilon)(x,\xi')$ such that $\omega(0)$ is real (since $\operatorname{Re} a(0) \neq 0$ by (1.2)).

Then we obtain the factorization

$$\xi_1^\ell + \tilde{a}h_\varepsilon = (\xi_1 + \omega)\sum_{j=1}^{\ell-1}(-\omega)^{j-1}\xi_1^{\ell-1}$$

Set $\omega_\lambda(y,\eta') = \omega(y_1,\lambda y',\lambda\eta')$ and set $\ell_{2,\lambda}(y,\eta) = \sum_{j=1}^{\ell-1}(-\lambda^{-2\delta}\omega_\lambda)^{j-1}\eta_1^{\ell-j}$ $(\delta = (\ell-1)/\ell)$. Since $\ell_{2,\lambda}(y,\eta)$ satisfies (2.27) with $m = \ell-1$ and $\delta = (\ell-1)/\ell$, we get (2.24) if we show for some C

$$\lambda^{-2\sigma}\|h_{\varepsilon,\lambda}(y',D_{y'})v\| \leq C(\|(D_{y_1} + \lambda^{-2\delta}\omega_\lambda(y,D_{y'}))v\| + \|v\|), \tag{2.29}$$

if $v \in S$ vanishes $|y_1| > \varepsilon$.

For brevity we denote $\tilde{a}(x,\xi')h_\varepsilon(x',\xi')$ by $\tilde{a}(x,\xi')$ in what follows. Note that $\operatorname{Re}\tilde{a}$ is independent of x_1 on account of (1.3) because H_{p_1} and Γ were transformed to ∂_{x_1} and $\xi_1 = 0$, respectively, by the χ. Using the expansion $(1+z)^{1/\ell} = 1 + z/\ell + O(z^2)$, we obtain

$$\begin{aligned}\omega(x,\xi') &= (\operatorname{Re}\tilde{a})^{1/\ell}(1 + \operatorname{Im}\tilde{a}/\operatorname{Re}\tilde{a})^{1/\ell} \\ &= r(x',\xi') + iq(x,\xi') + O(|x| + |\xi'|)q(x,\xi')\end{aligned} \tag{2.30}$$

if we set $r = (\operatorname{Re}\tilde{a})^{1/\ell}$, $q = \operatorname{Im}\tilde{a}/\ell(\operatorname{Re}\tilde{a})^{(\ell-1)/\ell}$.

Hence (2.29) follows if we show that for some C

$$\begin{aligned}&\lambda^{-2\sigma}\|h_{\varepsilon,\lambda}(y',D_{y'})v\| + \lambda^{-2\delta}\|q_\lambda(y,Dy')h_{\varepsilon,\lambda}(y',D_{y'})v\| \\ &\leq C(\|D_{y_1} + \lambda^{-2\delta}(r_\lambda(y',D_{y'}) + iq_\lambda(y,D_{y'}))v\| + \|v\|),\end{aligned} \tag{2.31}$$

if $v \in S$ vanishes for $|y_1| > \varepsilon$,

where $q_\lambda(y,\eta') = q(y_1,\lambda y',\lambda\eta')$ and $r_\lambda(y',\eta') = r(\lambda y',\lambda\eta')$. Indeed, if we take $h_{\varepsilon''}(x',\xi')$ such that $h_\varepsilon(x',\xi') = 1$ on supp $h_{\varepsilon''}$ and substitute $h_{\varepsilon'',\lambda}v$ into (2.31), then we get (2.29) with replaced v by $h_{\varepsilon'',\lambda}v$ since the part corresponding to third term of the left hand side of (2.30) can be estimated by the second term of the right hand side of (2.31) when ε is small enough. (Note that $\varepsilon'' < \varepsilon$).

Let $\Phi(x,\xi')$ be the soulution to

$$(2.32)\qquad \partial_{x_1}\Phi + r(x',d_x,\Phi) = 0\ , \qquad \Phi(0,x',\xi') = x'\xi'$$

Without loss of generality we assume that the Φ exists on $\{|x_1| < \varepsilon\} \times \text{supp } h_\varepsilon(x',\xi')$. Put $\Phi_\lambda(y,\eta') = \lambda^{-2}\Phi(\lambda^{2/\ell}y_1,\ y',\lambda\eta')$. Then Φ_λ of course exists on $\{|y_1| < \varepsilon\} \times \text{supp } h_{\varepsilon,\lambda}(y',\eta')$. If we regard y_1 as a parameter, in the same way as in (2.10) and (2.11), we can define the Fourier integral operator (and the conjugate Fourier integral operator wi phase function $\Phi_\lambda(y,\eta')$ and the symbol $f_\lambda(y,\eta')$ satisfying

$$(2.6')\qquad |\partial_y^\alpha\partial_{\eta'}^{\beta'}f_\lambda| \le C_{\alpha\beta'}\lambda^{|\alpha'+\beta'|}$$

by

$$(2.33)\qquad F_{\Phi_\lambda}v(y) = \int e^{i\Phi_\lambda(y,\eta')}k_\lambda(y',\eta')f_\lambda(y,\eta')\check{v}(y_1,\eta')d\eta'$$

and

$$(2.34)\qquad F_{\Phi_\lambda}^*v(y) = \iint e^{i(y'\eta'-\Phi_\lambda(y_1,\tilde{y}',\eta'))}k_\lambda(\tilde{y}',\eta')$$
$$f_\lambda(y_1,\tilde{y}',\eta')v(y_1,\tilde{y}')d\tilde{y}'d\eta',$$

when $v \in S$ vanishes for $|y_1| \ge \varepsilon$. Here $\check{v}(y_1,\eta')$ denotes the Fourier transform of $v(y)$ with respect to y'.

Set

(2.35) $\Psi_\lambda(y,\eta) = y_1\eta_1 + \Phi_\lambda(y,\eta')$.

Let χ_λ denote a canonical transformation with generating function Ψ_λ, that is, $\chi_\lambda;\ (y, d_y\Psi_\lambda(y,\eta)) \mapsto (d_\eta\Psi_\lambda(y,\eta),\eta)$. Note that χ_λ and χ_λ^{-1} are defined for $\{|y_1| < \varepsilon\} \times \operatorname{supp} h_{\varepsilon,\lambda}(y',\eta')$ if ε is small enough. Set $\tilde{q}_\lambda(y,\eta') = q_\lambda \circ \chi_\lambda(y,\eta')$. It is easy to check that

(2.36) $$\tilde{q}_\lambda(y,\eta') = q(y_1, \psi_\lambda(y,\eta'), d_x\Phi(\lambda^{2/\ell}y_1, \psi_\lambda(y,\eta'),\lambda\eta'))$$

where $\psi_\lambda(y,\eta') = \psi(\lambda y';\lambda^{2/\ell}y_1,\lambda\eta')$ and $\psi(x';x_1,\xi')$ is defined as the inverse of $x' = d_\xi\Phi(x_1,\cdot,\xi')$.

Lemma 2.11. Assume that for some $\varepsilon_1 > 0$ and some constant C_1 the estimate

(2.37) $$\lambda^{-2\sigma}\|h_{\varepsilon_1,\lambda}(y',D_{y'})v\| + \lambda^{-2\delta}\|\tilde{q}_\lambda(y,D_{y'})h_{\varepsilon_1,\lambda}(y',D_{y'})v\| \le C_1(\|(D_{y_1} + i\lambda^{-2\delta}\tilde{q}_\lambda(y,D_{y'}))v\| + \|v\|)$$

holds if $v \in S$ vanishes for $|y_1| > \varepsilon_1$. Then (2.3) holds for some $\varepsilon > 0$ and C.

Proof. Fix y_1 as a parameter and let $\Phi_\lambda(y_1,y',\eta')$ correspond to $S_\lambda(y,\eta)$ in Definition 2.4. By the remark after Corollary 2.8, we see that the $\lambda^{-2}(I^*_{\Phi_\lambda}q_\lambda - \tilde{q}_\lambda I^*_{\Phi_\lambda})$ as an operator on $L_2(R^{n-1}_{y'})$ has a uniform bound with respect to $|y_1| < \varepsilon$ and $0 < \lambda \le 1$, therefore it has a uniform bound as an operator on $L_2([-\varepsilon,\varepsilon] \times R^{n-1}_{y'})$ by intergrating with respect to y_1.

Note that

$$D_{y_1} I^*_{\Phi_\lambda} v = I^*_{\Phi_\lambda} D_{y_1} v - (\partial_{y_1} \Phi_\lambda)^* {}_{\Phi_\lambda} v$$

$$= I^*_{\Phi_\lambda} D_{y_1} v + \lambda^{-2\delta} T^*_{\Phi_\lambda} v \quad ,$$

if $v \in S$ vanishes for $|y_1| > \varepsilon$,

where $t_\lambda(y,\eta') = r_\lambda(y', d_y, \Phi_\lambda(y,\eta'))$. In fact this follows from $\partial_{y_1} \Phi_\lambda(y,\eta') = -\lambda^{-2\delta} r_\lambda(y', d_y, \Phi_\lambda(y,\eta'))$. The adjoint form of Proposition 2.7 shows that the second term equals $I^*_{\Phi_\lambda} \lambda^{-2\delta} R_\lambda$ modulo L_2 - bounded operator. Hence, substitution $I^*_{\Phi_\lambda} v$ for v of (2.37) gives (2.31) because for $\tilde{h}_{\varepsilon,\lambda}(y_1,y',\eta')$ defined from $h_\varepsilon(x',\xi')$ in the same way as (2.36), there exists some ε' such that $\tilde{h}_{\varepsilon,\lambda} = 1$ on supp $h_{\varepsilon'}(\lambda y', \lambda\eta') \times \{|y_1| < \varepsilon\}$, provided that $0 < \lambda \leqq \lambda_0$ for some sufficiently small λ_0 .

In the rest of this section we investigate the properties of $\tilde{q}_\lambda(y,\eta')$ derived from the assumptions. It follows from (2.36) that for any α, β' and some $C_{\alpha\beta'}$ independent of λ

$$|\partial_y^\alpha \partial_\eta^{\beta'} \tilde{q}_\lambda(y,\eta')| \leqq C_{\alpha\beta'} \lambda^{|\alpha'+\beta'|} \tag{2.38}$$

on $\Omega_\varepsilon = \{|y_1| < \varepsilon \ , \ |y'| + |\eta'| < \varepsilon\lambda^{-1}\}$.

The second property is that

(2.39) $\tilde{q}_\lambda(y,\eta')$ does not change sign from + to - for y_1 increasing if λ is small enough.

This follows from (1.10). In fact, since it follows from (2.25) that

$$d_{x,\xi}(\operatorname{Re} a_\gamma \circ \chi)^{1/\ell})(x,0,\xi') = (d_{x,\xi}(\operatorname{Re} a_\gamma \circ \chi)^{1/\ell})(x,0,\xi')$$

we obtain the property (2.39) because the modified-null-bicharacteristic curve is invariant under canonical transformations χ and χ_λ. The invariance of Poisson brackets for canonical transformations and (2.25) give the following;

$$(2.40)\quad \begin{cases} \text{for any } (y,\eta') \in \Omega_\varepsilon = \{|y_1| < \varepsilon, |y'|+|\eta'| < \varepsilon\lambda^{-1}\} \\ \text{there exists a } I \in \mathcal{J}_\mu \text{ such that } \tilde{p}_{I,\lambda}(y,\eta') \not\equiv 0 \\ \text{if } \lambda \text{ is small enough.} \end{cases}$$

Here $\tilde{p}_{I,\lambda}(y,\eta')$ is defined in the same way as $p_I(x,\xi)$ of of (1.11) with $p_1 = \eta_1$ and $p_2 = \lambda^{-2\delta}\tilde{q}_\lambda(y,\eta')$. Indeed, note that (1.12) is invariant under canonical transformations and in changing p_i ($i = 1,2$) to $f_i p_i$ for non-vanishing functions f_i ($i = 1,2$). By means of (2.25) and the definition of $\mathcal{J}$, we see that for any $(y,\eta') \in \Omega_\varepsilon$ there exist a $I \in \mathcal{J}_\mu$ and $c_I > 0$ such that

$$|p^0_{I,\lambda}(y,\eta')| \geqq c_I\lambda^{-2+2b(I)/\ell} \quad ,$$

where $p^0_{I,\lambda}$ is defined by setting $p_1 = \eta_1$ and $p_2 = \lambda^{-2\delta}q_\lambda(y,\eta')$. If $p_{I,\lambda}$ denotes $p^0_{I,\lambda}$ with replaced η_1 by $\eta_1+\lambda^{-2\delta}r_\lambda(y',\eta')$, we obtain

$$|p_{I,\lambda} - p^0_{I,\lambda}| \leqq C_I\lambda^{-2\delta+2b(I)/\ell}$$

for some constant C_I determined by the derivatives of r and q. Consequently, the invariance of Poisson brackets under χ_λ gives (2.40).

3. Proof of Theorem 1.1.

In order to prove Theorem 1.1, as obserbed in the preceding section, it suffices to show (2.37).For the sake of simplicity, we denote $\lambda^{-2\delta}\tilde{q}_\lambda(y,\eta')$ by $q(x,\xi')$. Suppose that λ is small enough. By means of a constant scale change in the variables, we may assume from (2.38)-(2.40) that $q(x,\xi')$ satisfies the following conditions: For any α,β and some constant $C_{\alpha\beta}$ (independent of λ)

$$(3.1)\qquad |D_\xi^\alpha D_x^\beta q(x,\xi')| \leqq C_{\alpha\beta}\lambda^{|\alpha'+\beta'|-2\delta}$$

$$\text{in}\quad \Omega = \{|x_1| < 1,\ |(x',\xi')| < \lambda^{-1}\}$$

(3.2) $q(x,\xi')$ does not change sign from + to - for x_1 increasing.

(3.3) For any $(x,\xi') \in \Omega$ there exist some $\mu \in Q_0$ and some $c_0 > 0$ such that

$$\sum_{I\in J_\mu} \lambda^{2-2b(I)/\ell}|p_I(x,\xi')| \geqq c_0 > 0$$

provided that $p_1 = \xi_1$, $p_2 = q(x,\xi')$. Then (2.37) is stated as follows; for some C

$$(3.4)\qquad \lambda^{-2\sigma}\|h(\lambda x',\lambda D')u\| + \|D_{x_1}h(\lambda x',\lambda D')u\|$$

$$\leqq C(\|(D_{x_1} + iq(x,D'))u\| + \|u\|),$$

$$\text{if}\quad u \in S \quad \text{vanishes for} \quad |x_1| > 1/2\ ,$$

where h is C_0^∞ with support in a ball of radius $1/2$.

The proof of (3.4) for $q(x,\xi)$ with (3.1)-(3.3) is the same as in showing [4,(6.1) and (6.30)] except the difference of "weight". (See [4,(4.1) and (4.3)].) To prove [4, (6.1) and (6.30)] it was important to obtain inequalities for q in [4,Section 4]. So we sketch the argument corresponding to [4,Section 4]. Put

$$M(x,\xi') = \max_{I \in J_\mu} |p_I(x,\xi')/\rho|^{1/|I|} . \tag{3.5}$$

Here ρ is a large parameter, whose role is the same as in [4,Section 4] (see [4, p.149]). By (3.1) and (3.3) we have

$$C_1\lambda^{-2\delta/\mu}\ \rho^{-(\ell-1)/\mu} \leqq M(x,\xi') \leqq C_2\lambda^{-2\delta}/\rho . \tag{3.6}$$

Here and in what follows the constants are independent of λ and ρ .

The definition of $M = M(0)$ means in particular that

$$|D_{x_1}^j q(0)| < \rho M^{j+1} \quad , \quad j \leqq \mu-1 \quad , \tag{3.7}$$

and where (3.1) is valid we have by (3.6)

$$|D_{x_1}^j q| \leq O(\lambda^c)\rho M^{[\mu]+1} \tag{3.8}$$

since $\rho^{\ell-2}/M^{[\mu+1]-\mu} << 1$ if λ is small enough, where c is some positive. If we set

$$F(t,y',\eta') = q(t/M,\ y'(\rho M)^{1/2\delta},\eta'(\rho M)^{1/2\delta})/\rho M \quad ,$$

then the application of [4,Lemma 7.1] to $F(t,y',\eta')$ shows that

$$(3.9)\qquad |D_\xi^\alpha D_x^\beta (\tilde{q}(x,\xi') - \xi_2 \partial\tilde{q}(x_1,0)/\partial\xi_2)|$$

$$\leqq C_{\alpha\beta}\rho M^{\beta_1+1}(\rho M)^{-|\alpha'+\beta'|/2\delta}$$

$$\text{if}\quad |x_1 M| < 1\ ,\ |(x',\xi')| < C(\rho M)^{1/2\delta}\ ,$$

where $\tilde{q}$ is determined from q by a sympletic orthogonal transfomation.

Let $\varepsilon = \lambda^\kappa$ with $0 < \kappa < 1/\mu$ (which is different from ε in the preceding section). Then

$$(3.10)\qquad \varepsilon^2(\rho M)^{1/\delta} >> 1\ .$$

As in [4], using this ε we consider the following two cases.

Case I. Assume that

$$(3.11)\qquad |d_{x',\xi'} D_{x_1}^j q(0)| \leqq \varepsilon\rho M^{j+1}\ ,\ j < \mu'\ .$$

where $\mu' = (\ell-2)(\mu-1)/2(\ell-1)$. In view of (3.1), (3.6) and (3.10) we get for some $c > 0$

$$(3.11)'\qquad |d_{x',\xi'} D_{x_1}^j q(x,\xi')| \leq O(\lambda^c)\varepsilon\rho M^{[\mu']+1}\ ,$$

$$j > \mu'\ ,\quad (x,\xi') \in \Omega\ .$$

Then it is easy to check that the argument corresponding to Case I in [4,Section 4] follows with k and $k/2$ replaced by $[\mu-1]$ and μ'.

Case II. Assume now that (3.11) is not fulfilled. Choose $s < \mu'$ so that with $q^{(j)} = \partial^j q/\partial x_1^j$

$$d_{x',\xi',q^{(s)}}(0) = a \tag{3.12}$$

$$d_{x',\xi',q^{(j)}}(0) \leq aM^{j-s} \quad \text{for} \quad j < \mu' \tag{3.13}$$

Then (3.1) and the fact that (3.11) is not valid give

$$\varepsilon\rho M^{s+1} < a \leq C\lambda^{1-2\delta} \ . \tag{3.14}$$

In view of (3.11)' we have then for every j

$$|D_{x_1}^j \tilde{q}(x_1,0)/\partial\xi_2| \leq C_j aM^{j-s} = C_j\rho M^{j+1}A_2 \ , \tag{3.13)'}$$

$$|x_1 M| < 1 \ ,$$

where $A_2 = a/\rho M^{s+1}$. The equality of (3.13)' holds when $j = s$. From (3.9) we can therefore obtain an estimate of the form [4,(4.10)] with $B_2 = A_3 = B_3 = \ldots = (\rho M)^{-1/2\delta}$ and $K = M$. However, $A_2B_2 = a/M^{s+1}(\rho M)^{1/2\delta}$ so [4, Lemma 4.1] is not applicable if $a > M^{s+1}(\rho M)^{1/2\delta}$. In that case we shall replace the orthogonal sympletic transformation which led from q to $\tilde{q}$ by a non-linear canonical transformation.

Thus assume for the moment that (3.12), (3.13) and

$$M^{s+1}(\rho M)^{1/2\delta} < a \leq C\lambda^{1-2\delta} \tag{3.14)'}$$

are fulfilled. Let $b = a\lambda^{2(\delta-1)}$. Then the function

$$Q(x',\xi') = (\, q^{(s)}(0,bx',b\xi') - q^{(s)}(0,0,0))/ab$$

is in a bounded subset of $C^\infty(U)$ ($U = \{|(x',\xi')| < C^{-1}\}$) since $|(bx',b\xi')| < \lambda^{-1}$. Hence there exists some canonical transformation χ belonging to a bounded set in C^∞ for $0 < \lambda \leqq 1$ such that

$$Q\circ\chi(x',\xi') = \xi_2$$

in a neighborhood of 0. If we put

$$\chi_b(x',\xi') = b\chi(b^{-1}x',b^{-1}\xi')$$

$$\tilde{q}(x,\xi') = q(x_1,\chi_b(x',\xi')) \ ,$$

then we obtain

$$\tilde{q}^{(s)}(x,\xi') = a\xi_2 + \tilde{q}^{(s)}(0,0) \tag{3.15}$$

when $x_1 = 0$, $|(x',\xi')| < cb$.

By the same way as in [4] we get

$$|a^j(\partial/\partial x_2)^j(\partial/\partial x_1)^i\tilde{q}(0)| \leqq C_{ij}M^{j(s+1)}\rho M^{i+1} \tag{3.16}$$

for any i, j satisfying

$$j \leqq \ell - 2 \quad \text{and} \quad (i+1+j(s+1))(\ell-1)/(\ell-j-1) \leqq \mu \ .$$

If we introduce

$$B_2 = M^{s+1}/a \ , \tag{3.17}$$

noting incidentally that $\rho A_2B_2 = 1$ as required in [4,Lemma 4.1], this means that we have bounds for the derivatives of $\tilde{q}(x_1/M,\ x_2/B_2,\ 0)/\rho M$ at 0 .

As stated in [4, p.156], when we derive (3.16), we can replace the canonical transformation $\chi_b(x',\xi')$ by another $\tilde{\chi}_b(x',\xi')$ which is linear in all variables except x_2, provided that the integral curve of the Hamilton field of $\tilde{q}^{(s)}(0,x',\xi')$ by new $\tilde{\chi}_b$ is also the x_2 axis $x_2 = at$, $x_3 = \dots = \xi_n = 0$. We denote $\tilde{\chi}_b$ by χ_b in what follows.

By the analogous calculation as in showing [4,(4.23) and (4.23)'] we obtain when $|x_1 M| < 1$, $|(x',\xi')| < cb$

$$(3.18)\quad |D_\xi^\alpha D_x^\beta \tilde{q}(x,\xi')| \leqq C_{\alpha\beta} a M^{\beta_1} b^{1-|\alpha'+\beta'|} \text{ if } |\alpha'+\beta'| \neq 0$$

$$(3.18)'\quad |D_\xi^\alpha D_x^\beta \tilde{q}(x,\xi')| \leqq C'_{\alpha\beta} \lambda^{-2\delta} b^{-|\alpha'+\beta'|} \quad \text{for any } \alpha',\ \beta'.$$

In particular (3.18)' is a much better estimate than (3.16) if $j \geqq \ell-1$ or $(i+1+j(s+1))(\ell-1)/(\ell-j-1) > \mu$, and it is not only valid at 0. Hence (3.16) leads us to uniform bounds for $\tilde{q}(x_1/M, x_2/B_2, 0)/\rho M$ and all of its derivatives when $|x_1| < 1$ and $|x_2| < 1$.

Since $b >> (\rho M)^{1/2\delta}$ we can apply [4. Lemma 7.1] to

$$F(t,x_2,y) = (M\rho)^{-1} \tilde{q}(t/M, x_2/B_2, x''(\rho M)^{1/2\delta}, \xi'(\rho M)^{1/2\delta})$$

where $x'' = (x_3,\dots,x_n)$ and $y = (x'',\xi')$. Therefore, by the same way as in getting [4,(4.25)] we obtain

$$(3.19)\quad |D_\xi^\alpha D_x^\beta (\tilde{q}(x,\xi') - \xi_2 \partial\tilde{q}(x_1,x_2,0)/\partial\xi_2)$$

$$\leqq C M^{\beta_1+1} \rho B_2^{\beta_2} (\rho M)^{-|\alpha'+\beta'|/2\delta}$$

$$\text{when } |x_1|M < 1,\ |(x'',\xi')| < (\rho M)^{1/2\delta},\ |x_2| < 1/B_2.$$

As in [4], we obtain [4, (4.26)] and [4, (4.27)] with replaced the right hand side by $Cb^{-\beta}2\lambda^{2/\ell}$. Hence it follows from (3.12) that $(B_2/M)\partial\tilde{q}(x_1/M,0)/\partial\xi_2$ is essentially a normallized polinomial in x_1 of degree $[\mu']$ and $\partial\tilde{q}(x_1/M,x_2,0)/\partial\xi_2$ is almost independent of x_2 .

From (3.19) and these inequalities we obtain with $B_1 = M$, $A_j = B_j = (\rho M)^{-1/2\delta}$ when $j > 2$

$$(3.20)\qquad |D_\xi^\alpha D_x^\beta \tilde{q}(x,\xi')| \leq C_{\alpha\beta}\rho M A^\alpha B^\beta \text{ , if } |x_1 M| < 1,\ |x_2 B_2| < 1,$$
$$|\xi_2 A_2| < N,\ |(x'',\xi'')| < (\rho M)^{1/2\delta} ,$$

where N is a fixed but arbitrary constant. If we write $\tilde{M}(x,\xi') = M(x_1,\chi_b(x',\xi'))$, it follows in view of [4,Lemma 4.1], where we take $A_1 = 1/M$ and $K = M$, that

$$(3.21)\qquad \tilde{M}(x,\xi') \leq C_N M \text{ if } |x_1 M| < 1 ,\ |x_2 B_2| < 1 ,$$
$$|\xi_2 A_2| < N,\ |(x'',\xi'')| < (\rho M)^{1/2\delta} .$$

when (3.14) is fulfilled but not (3.14)' we get the same conclusion with B_2 replaced by $(\rho M)^{-1/2\delta}$ and χ_b equal to orthogonal sympletic transformation such that $\tilde{q}(x,\xi') = q(x_1,\chi_b(x',\xi'))$.

The argument corresponding to the rest of [4, Section 4] can be done by the same way if we let (3.1)-(3.3), (3.16), (3.18), (3.18)', (3.19), (3.20), (3.21), $[\mu-1]$, μ' , χ_b and

$(\rho M)^{1/2\delta}$ correspond to [4, (4.1)-(4.3), (4.21), (4.23), (4.23)', (4.25), (4.28), (4.29), k, k/2, χ_a and $\sqrt{\rho M}$] respectively.

Because we have got the result corresponding to [4,Section 4] we can easily prove (3.4) by the same way as in [4, Section 6], if we take the above correspondence. The detail is left to the reader.

4. Proof of Proposition 1.2.

The method of the proof is a version of [2] (, see also [15]). Suppose that $\tilde{L}$ is hypoelliptic at the origin Then, as well-known, there exist a positive integer s, a constant C and some neighborhood U of 0 such that

$$(4.1) \qquad \|u\| \leq C\| \Lambda^s \tilde{L} u\| \quad , \ u \in C_0^\infty(U) \ ,$$

where the symbol of Λ is $(|\xi_1|^{10} + |\xi'|^8 + 1)$. Hence for any large N there exists a C_N such that

$$(4.1)' \qquad \|u\| \leq C_N(\| \Lambda^s \tilde{L} u \| + \| |x|^N u \| + \| |x|^N \Lambda^s \tilde{L} u \|),$$

$$u \in S \ .$$

If we take the canonical change of variables (x,ξ) into

$$(4.2) \qquad (\lambda^2 x_1, \lambda^9 x', \lambda^{-2\tau-2}(\xi_{01} + \lambda^{2\tau}\xi_1), \lambda^{-2\tau-9}(\xi_0' + \lambda^{2\tau}\xi')),$$

$$(0 < \lambda \leq 1 \ , \ \tau = 13),$$

with a fixed $\xi_0 = (\xi_{01}, 0, \xi_{03})$ satisfying

$$\xi_{01}^5 + 2\xi_{03}^4 = 0 \quad \text{and} \quad \xi_{03} > 0 \ ,$$

then it follows from (4.1)' that

$$(4.3) \qquad \| u \| \leq C_N'(\lambda^{-M}\| L_\lambda u \| + \lambda^{2N-M} \| R_0(x,\lambda^{2\tau}D_x)u\|),$$

$$u \in S \ ,$$

where

$$L_\lambda = L_1(\lambda^{2\tau}D_x) + i\lambda^\tau x_1^2 x_2 \xi_{03}^4 + \sum_{j=1}^{2} \lambda^{\tau(2-j)} R_j(x,\lambda^{2\tau}D_x),$$

$$L_1(\xi) = 5\xi_{01}^4 \xi_1 + 4\xi_{03}^3 (2\xi_3 - \xi_2) ,$$

$$R_j(x,\xi) = \sum_{\substack{j \leqq |\alpha| \\ |\alpha+\beta| \leqq N_j}} a_{\alpha\beta j}(\lambda) x^\beta \xi^\alpha \qquad (j = 0,1,2)$$

$$(a_{\alpha\beta j}(\lambda) = O(\lambda^c) , \quad c \geqq 0).$$

Here M and N_j ($j=0,1,2$) are some integers depending on s . Since $\xi_{03} > 0$ we can take $w(x)$ such that

$$L_1(\partial_x w) + i x_1^2 x_2 \xi_{03}^4 = 0$$

(4.4)

$$|\mathrm{Im}\ w| \geqq c_0 |x|^4 , \quad c_0 > 0 .$$

Put $u_\lambda(x) = (\exp i\lambda^{-\tau} w(x)) \sum_{j=0}^{N} v_j(x,\lambda)\lambda^{\tau j}$, where v_j will be determined later such that they are polinomials of x whose coefficients have the same property as $a_{\alpha\beta j}$. Then we have

$$L_\lambda u_\lambda = \exp i\lambda^{-\tau} w \sum_{j=2}^{N} (L_1(D_x) + A(x,\partial_x w)) v_{j-2}$$

$$+ F_j(x)) \lambda^{\tau j} , \qquad (F_2 = 0).$$

Here $A(x,\xi) = \sum_{|\alpha|=1} a_{\alpha\beta 1} x^\beta \xi^\alpha$ and $F_j(x)$ is a linear combination of the functions $v_0,\ldots,v_{j-3}$ and their derivatives. By means of Cauchy-Kowalewska theorem, solve the transport equations

$$(L_1(D_x) + A(x,\partial_x w)) v_j + F_{j+2}(x) = O(|x|^{N-j})$$

under the condition $v_0(0,\lambda) = 1$ and $v_j(0,\lambda) = 0$ ($j \geqq 1$),

successively. Then the analytic solution obtained, which is defined in a certain neighborhood ω of the origin, is to be multiplied by a cut off function $\phi \in C_0^\infty(\omega)$ which equals 1 in another neighborhood of the origin. The multiplication does not affect (4.5) because $(1-\phi)v_j = \mathcal{O}(|x|^N)$. Substituting $u_\lambda(x) \in \mathcal{A}$ into (4.3) and changing variables x into $\lambda^{\tau/4}x$ give us the contradiction if $2N > M$ and if λ tends to 0 .

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NOM: MORIMOTO Prénom: Yoshinori

Sujet: La propagation des singularités et l'hypoellipticité pour les équations aux dérivées partielles

Résumé

Cette thèse est consacrée à l'étude des régularités et irrégularités des solutions des équations aux dérivées partielles. La première partie traite de la propagation du front d'onde des solutions du problème de Cauchy hyperbolique. Tous les articles de la première partie se rapportent au résultat de Kumano-go et Taniguchi sur la propagation des singularités pour un système hyperbolique d'ordre 1 à partie principale diagonale. Dans la deuxième partie on étudie l'hypoellipticité des opérateurs pseudo-différentiels à caractéristiques multiples.

Mots Clés :

Opérateur pseudo-différentiel, Opérateur integral de Fourier, du front d'onde, transformation canonique, trajectoire admissible.